Exact statistical moments of multi-mode stochastic hybrid system with renewal transition

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***Short Abstract* — In this paper, a class of stochastic hybrid systems comprising of multiple operation modes is studied. In each mode, the state evolves according to a linear stochastic differential equation. We allow for stochastic switching between operational modes with switching times controlled by an underlying renewal process such that the time spent in each mode is a random variable with an arbitrary given probability distribution. We present a novel method to derive exact analytical solutions for the statistical moments and illustrate the applicability of the method on an example drawn from systems biology. More specifically, we study how random switching of a gene between transcriptionally active and inactive states drives stochastic variation in the level of the expressed protein. Our results show that while randomness in gene switching times has no effect on the mean protein level, it critically impacts the magnitude of fluctuations in the protein level. This effect is further amplified for proteins with high decay rate. We finally discuss how noise in protein can be used to infer the underlying gene expression mechanisms.**

***Keywords* — stochastic hybrid system, gene switching noise**

I. Introduction

Stochastic hybrid systems consists of continuous evolution of states and discrete random events that change the values of states and/or continuous dynamics of the system These systems arise naturally in modeling of a wide range of applications ranging from systems biology to communication networks.[1] For instance, inside a living cell species are subject to gene duplication which can increase their synthesis rate up to two-fold (changing species dynamics). The next phase after duplication is cell division which reduces the synthesis rate to its value before duplication and also changes the counts of species to half in each daughter cell (changing the values of states). Given their applicability, a large set of analysis tools exists for these systems. For example, moment dynamics are used to obtain their statistical moments and forward Kolmogorov equation is used to obtain the probability density function of the states. However, these methods suffer from the caveat of un-closeness if there exists nonlinearity in the system. This means that finding probability density function or statistical moments would require solving a set of equations with infinite dimension. To address this problem, many approximation methods are proposed, for instance the technique of moment closure which provides approximated (typically numerical) results of the moments.

In this paper we provide a new method to derive the exact analytical solutions of the moments for a general class of stochastic hybrid systems. We provide the exact solutions of first and second order moments. However, our approach can be generalized to derive any arbitrary moment of these systems (skewness, kurtosis, etc.). We demonstrate our method on an example drawn from systems biology. We quantify noise in protein concentration in the presence of random gene switching times and random synthesis events. We observe that randomness in gene-switching time increases the total noise in protein concentration. Since the noise in gene switching time intervals is a function of the number of steps that needs to be taken before transcription starts, we discuss how noise in protein sheds light on the underlying gene expression mechanisms.

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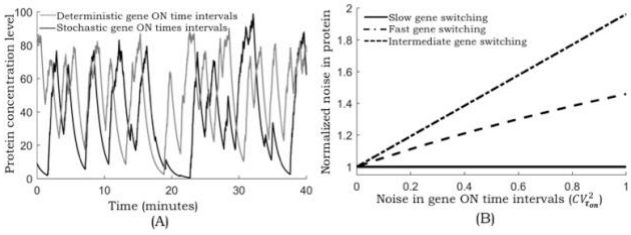
II. Main Results



The noise in protein concentration is highly affected by the gene-switching time intervals. Noise in gene ON time intervals will not change the mean of a protein. Moreover, it does not have an obvious effect on the protein time trend. The noise in protein is highly affected by the gene switching noise. While the mean of protein only depends on the ratio of gene ON and OFF times, noise is affected by the magnitude of the ON and OFF time intervals as well

III. Summary

We studied statistical moments of a class of stochastic hybrid systems with multiple operation modes. We derived exact solution of the first and second order moments as well as necessary and sufficient conditions for having finite moments. While we only present our derivations for a two mode system, the results can be generalized to a any arbitrary number of modes.

References

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