Troisième cycle de la Physique en Suisse Romande

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Field Theory and Nonequilibrium Statistical Mechanics

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Contents

Chapter 1

Introduction

In this course, we shall be concerned with the time-dependent behaviour of systems close to a critical point. These may be *equilibrium* (or close to equilibrium) systems, *or* systems maintained in/close to some steady-state which is not equilibrium, by some driving force.

These will be the two main parts of the course. However, it will emerge that many of the *scaling properties* of such systems are similar, whether or not they are in equilibrium. As a result, the most effective way of understanding these, the *renormalisation group* (RG) and dynamic field theory are very similar.

We shall restrict ourselves to systems at **finite temperature**, which turns out to mean that, in the critical region, the thermal fluctuations are more important than the quantum ones. Thus, the system is in contact with a **heat bath** which, in the absence of driving force, will produce dissipation and relaxation toward equilibrium.

Hence, the effective equations of motion we shall use have a direction of time built into them. This is not to say that no features of the underlying time reversal invariant dynamics remain : for example, any *conservation laws* in the full dynamics should also be respected by the effective equations.

Conservation laws $\rightarrow$ slow modes
$\rightarrow$ affect long-time dependence $(\omega, k \rightarrow 0)$.

Note that there are other mechanisms for producing slow modes, e.g. Goldstone bosons, which arise from the spontaneous breaking of a continuous symmetry.

Part I

Critical dynamics near equilibrium phase transitions

Chapter 2

Basic Principles

In dynamic critical behaviour, there are different kinds of observable quantities. Consider a magnetic system with $s(r,t)$ being the local, time-dependent magnetisation.

2.1 Correlation functions

$$C(r-r',t-t') \equiv \langle s(r,t)s(r',t')\rangle \tag{2.1}$$

(in equilibrium) where

$$\langle s(r,t)s(r',t')\rangle \equiv \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} dt''\, s(r,t+t'')s(r',t'+t'') \tag{2.2}$$

$\langle s(r,t)s(r',t')\rangle$ is the *static* correlation function, and may be calculated by the usual equilibrium statistical mechanics formula :

$$\langle s(r,t)s(r',t)\rangle = \frac{1}{Z}Tr\left\{s(r)s(r')e^{-\beta\mathcal{H}}\right\} \tag{2.3}$$

2.2 Response functions

We may add a time-varying field $h(r,t)$ ($\mathcal{H} \longrightarrow \mathcal{H} - \sum_r h(r,t)s(r,t)$) which couples to $s(r,t)$ in the hamiltonian, and measure the *response* $\langle s(r,t)\rangle$. The *linear* response must have the form

$$\langle s(r,t)\rangle = \int G(r-r',t-t')h(r',t')\, d^d r' dt' \tag{2.4}$$

which defines G. Note that $G=0$ if $t<t'$ by causality.

2.3 Fluctuation-dissipation relation

C and G are related by

$$C(t-t') = k_B T \int_{-\infty}^{t'} G(t-t'')dt'' \qquad (t > t') \tag{2.5}$$

Let us see where this comes from for an Ising system in which $s(t) = \pm 1$ (we suppress the r-dependence for clarity). We have in equilibrium :

$$\langle s(t)s(t')\rangle = \frac{1}{2}\langle s(t)\rangle_{s(t')=+1} - \frac{1}{2}\langle s(t)\rangle_{s(t')=-1} \tag{2.6}$$

where $\langle s(t)\rangle_{s(t')=+1}$ means then conditional expectation value of $s(t)$, including only those histories when $s(t') = +1$.
Now imagine switching on a small field h at $t = -\infty$ and switching it off at $t = t'$. At that point the system will be in equilibrium in the presence of the field h, so the probability that $s(t') = \pm 1$ is :

$$\frac{\exp(\pm h/k_B T)}{2\cosh(h/k_B T)} \approx \frac{1}{2}\left(1 \pm \frac{h}{k_B T} + O(h^2)\right) \tag{2.7}$$

Hence

$$\begin{aligned}
\langle s(t)\rangle &= \frac{1}{2}\left(1+\frac{h}{k_B T}\right)\langle s(t)\rangle_{s(t')=+1} + \frac{1}{2}\left(1-\frac{h}{k_B T}\right)\langle s(t)\rangle_{s(t')=-1} \\
&= h\int_{-\infty}^{t'} G(t-t'')\,dt''
\end{aligned} \tag{2.8}$$

and (2.5) follows by equating terms $\mathcal{O}(h)$ (note that the $\mathcal{O}(1)$ terms cancel by symmetry).

Problem : Show for this simple model that the *nonlinear* response is also related to $C(t-t')$.

The FDT is usually expressed in terms of frequency space :

$$\tilde{G}(\omega) = \int_{-\infty}^{+\infty} dt\, G(t)e^{i\omega t} \qquad G(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\tilde{G}(\omega)e^{-i\omega t} \tag{2.9}$$

$$\tilde{C}(\omega) = \int_{-\infty}^{+\infty} dt\, C(t)e^{i\omega t} \qquad C(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\tilde{C}(\omega)e^{-i\omega t} \tag{2.10}$$

$$C'(t-t') = k_B T[G(t-t') - G(t'-t)] \tag{2.11}$$

from which we get

$$\tilde{C}(r,\omega) = \frac{2k_BT}{\omega} Im(\tilde{G}(r,\omega)) \tag{2.12}$$

$\Big[$ NB : This is the $\hbar \to 0$ limit of the *quantum* FDT :

$$\tilde{C} = 2\hbar \coth\left(\frac{\hbar\omega}{k_BT}\right) \mathrm{Im}(\tilde{G}) \tag{2.13}$$

which may be derived using Fermi's golden rule (see Landau & Lifshitz).$\Big]$

The RHS of equations (2.12),(2.13) is related to the *dissipation* : the energy is proportional to $-\sum_{r,r'} s(r)s(r')\delta(r-r')$, thus

$$dE/dt \propto \sum_{r,r'} \langle s(r,t)\dot{s}(r',t)\rangle \delta(r-r')$$

but

$$\langle s(r,t)\dot{s}(r',t)\rangle \equiv C'(0) \propto \int d\omega \, \mathrm{Im}\tilde{G}(\omega)$$

Thus $\mathrm{Im}\tilde{G}(\omega)$ gives the rate of energy dissipation power spectrum.

FDT follows from very general principles, and any effective description should respect it.

Chapter 3

Models of critical dynamics

3.1 Master equation

3.1.1 Definition

This is an equation of motion for the time evolution of the probability $P(\alpha, t)$ of finding a system in a microstate α. It has the form :

$$\frac{d}{dt}P(\alpha,t) = \sum_\beta R_{\beta\to\alpha}P(\beta,t) - \sum_\beta R_{\alpha\to\beta}P(\alpha,t) \tag{3.1}$$

The model determines the *rates* $R_{\alpha\to\beta}$.

Note that the probability is conserved : $\frac{d}{dt}\sum_\alpha P(\alpha,t) = 0$.

If this is supposed to describe the relaxation towards equilibrium, the Gibbs distribution $P(\alpha) \propto \exp(-E(\alpha)/kT)$ must be a steady-state solution. This means that

$$\sum_\beta \left[R_{\beta\to\alpha}e^{-E(\beta)/kT} - R_{\alpha\to\beta}e^{-E(\alpha)/kT}\right] = 0 \tag{3.2}$$

This will certainly be satisfied if the $[\cdot] = 0$ for *each* β (*detailed balance* condition). It requires

$$\frac{R_{\alpha\to\beta}}{R_{\beta\to\alpha}} = e^{-(E(\beta)-E(\alpha))/kT} \tag{3.3}$$

There are many solutions of this constraint, e.g.

$$R_{\alpha\to\beta} \propto \frac{e^{+\frac{1}{2}(E(\alpha)-E(\beta))/kT}}{e^{+\frac{1}{2}(E(\alpha)-E(\beta))/kT} + e^{-\frac{1}{2}(E(\alpha)-E(\beta))/kT}} \tag{3.4}$$

As $T \to 0$ we have *zero-temperature dynamics* :

$$R_{\alpha\to\beta} = \begin{cases} 1 & \text{if} \quad E(\beta) < E(\alpha) \\ \frac{1}{2} & \text{if} \quad E(\beta) = E(\alpha) \\ 0 & \text{if} \quad E(\beta) > E(\alpha) \end{cases} \tag{3.5}$$

Problem : Show that the Metropolis algorithm satisfies detailed balance.

3.1.2 Example : the Glauber model

An example of a master equation is given by the Glauber dynamics for the Ising model : Let us denote $s_1, s_2, ...$ the spins and $\alpha = \{s\}$ the microstates. The allowed transitions $\alpha \to \beta$ correspond to flipping a single spin :

$$R_j(\uparrow\to\downarrow) \qquad R_j(\downarrow\to\uparrow) \tag{3.6}$$

These rates will satisfy detailed balance if

$$\frac{R_j(\uparrow\to\downarrow)}{R_j(\downarrow\to\uparrow)} = \frac{e^{-h_j/kT}}{e^{+h_j/kT}} \tag{3.7}$$

where h_j is the local field caused by either the applied field or the other spins.

A solution is to take

$$R_j(\uparrow\to\downarrow) = \Gamma \frac{e^{-h_j/kT}}{e^{-h_j/kT} + e^{+h_j/kT}} \tag{3.8}$$

where Γ is a *rate* with dimensions $(\text{time})^{-1}$.

For example, for the one-dimensional Ising model, the allowed local processes, with their respective rates are, in the absence of an applied field ($H_{ext} = 0$) :

$\uparrow\uparrow\uparrow \quad \to \quad \uparrow\downarrow\uparrow$	$\Gamma \frac{\exp(-2J/kT)}{\exp(2J/kT)+\exp(-2J/kT)}$
$\uparrow\downarrow\uparrow \quad \to \quad \uparrow\uparrow\uparrow$	$\Gamma \frac{\exp(+2J/kT)}{\exp(2J/kT)+\exp(-2J/kT)}$
$\uparrow\uparrow\downarrow \quad \leftrightarrow \quad \uparrow\downarrow\downarrow$	Γ

Problem : Show that C & G calculated in the Glauber model satisfy the FDT.

These processes are more simply understood in terms of *domain walls.* The last processes correspond to random walks, or *diffusion* of domain walls. Their density ρ changes by the first two processes, and we can write :

$$\begin{aligned}\frac{d\rho}{dt} &= -2\Gamma\frac{e^{2J/kT}}{e^{2J/kT}+e^{-2J/kT}}\,\rho^2 \\ &\quad +2\Gamma\frac{e^{-2J/kT}}{e^{2J/kT}+e^{-2J/kT}}\end{aligned} \tag{3.9}$$

Thus at late times, $\rho \to \rho^* = e^{-2J/kT}$ which is just the correlation length ξ^{-1} in equilibrium.

The relaxation time (time for a single spin to flip) is $\approx$ to the time for a domain wall to diffuse a correlation length which is of the order of ξ^2. (Note that this is different from the relaxation time for ρ, which scales like $1/\rho^* \sim \xi$).

3.2 Langevin-type equation

This is a stochastic differential equation designed to generate the required distribution. It works better for systems with *continuous* degrees of freedom. The prototype is *Brownian motion* :

Consider a Brownian particle of unit mass. The equation of motion for the velocity (in 1-d) is :

$$\dot{v}(t) = F(t) - \Gamma v(t) + \zeta(t) \tag{3.10}$$

with :

$$\begin{aligned}F(t) &= \text{driving force} \\ \Gamma v(t) &= \text{friction} \\ \zeta(t) &= \text{random noise due to collisions}\end{aligned}$$

The dissipative term may be written :

$$-\Gamma\partial_v\left(\mathcal{H} = \frac{1}{2}v^2, \text{the energy}\right) \tag{3.11}$$

The noise is correlated only over times between microscopic collisions. Over longer times, we can therefore write :

$$\langle\zeta(t)\zeta(t')\rangle = 2D\delta(t-t') \tag{3.12}$$

where D is a constant. Its value is determined by the requirement that the steady-state distribution is Maxwellian, i.e. $\langle v^2 \rangle = kT$.

Integrating over a time interval dt yields :

$$v(t+\delta t) \approx (1-\Gamma\delta t)v(t) + \int_t^{t+\delta t} \zeta(t')dt' \qquad (3.13)$$

Note that both terms in the sum are uncorrelated, hence :

$$\langle v^2(t+\delta t)\rangle \approx (1-2\Gamma\delta t)\langle v^2(t)\rangle + 2D\delta t \qquad (3.14)$$

and finally

$$D = \Gamma kT \qquad \text{(Einstein relation)} \qquad (3.15)$$

NB : From the Langevin equation, we can also derive the *Fokker-Planck* equation, which describes the time evolution of the probability distribution $P(v(t), t)$.

3.3 Models A and B

3.3.1 Definition

This is the simplest purely relaxational model of an Ising ferromagnet. We work in *reduced units*, so $kT_c = 1$, $\Gamma = D$. The reduced Landau-Ginzburg hamiltonian is :

$$\mathcal{H} = \int \left[\frac{1}{2}(\nabla S)^2 + V(S)\right] d^d x \qquad (3.16)$$

where $V(S) = \frac{1}{2}r_0 S^2 + \frac{1}{4}uS^4$, $\quad r_0 \propto T - T_{MF}$.

Model A is :

$$\partial_t S(x,t) = -D\frac{\delta\mathcal{H}}{\delta S(x,t)} + \zeta(x,t) \qquad (3.17)$$

where $\frac{\delta\mathcal{H}}{\delta S} = -\nabla^2(S) + V'(S)$.

The Einstein relation now has the form :

$$\langle \zeta(x,t)\zeta(x',t')\rangle = 2D\delta^{(d)}(x-x')\delta(t-t') \qquad (3.18)$$

This follows as for Brownian motion : we have :

$$S(x,t+\delta t) \simeq S(x,t) - D\delta t \cdot \frac{\delta\mathcal{H}}{\delta S(x,t)} + \int_t^{t+\delta t} \zeta(x,t')dt' \qquad (3.19)$$

so

$$
\begin{aligned}
&\langle S(x,t+\delta t)S(x',t+\delta t)\rangle - \langle S(x,t)S(x',t)\rangle \approx \\
&\quad -D\delta t\left[\left\langle S(x,t)\frac{\delta\mathcal{H}}{\delta S(x',t)}\right\rangle + \left\langle S(x',t)\frac{\delta\mathcal{H}}{\delta S(x,t)}\right\rangle\right] \\
&\quad + \int_t^{t+\delta t} dt' \int_t^{t+\delta_t} dt'' \langle\zeta(x,t')\zeta(x',t'')\rangle
\end{aligned}
\tag{3.20}
$$

But, in equilibrium : $\langle S(x,t)\frac{\delta\mathcal{H}}{\delta S(x',t)}\rangle = \delta^{(d)}(x-x')$ by functional integration by parts, and the left hand side vanishes.

In Model A, the total magnetisation $\int S\, d^d x$ is not conserved. But in some physical systems, it might be (e.g. S=order parameter for liquid-gas critical point, or a binary fluid). In that case, we have a continuity equation for S :

$$\partial_t S = -\vec{\nabla}\cdot\vec{J} \tag{3.21}$$

where $\vec{J}$ is a current. To be consistent, we should therefore take

$$D \longrightarrow -D'\nabla^2 \qquad \text{[Question : Why the minus sign ?]} \tag{3.22}$$

with $D' > 0$. The Einstein relation is then :

$$\langle\zeta(x,t)\zeta(x',t')\rangle = -2D'\nabla^2\delta^{(d)}(x-x')\delta(t-t') \tag{3.23}$$

or equivalently, we can think of the noise term as being $\vec{\nabla}\cdot\vec{\zeta}$, in which case

$$\langle\zeta_i(x,t)\zeta_j(x',t')\rangle = -2D'\delta_{i,j}\delta^{(d)}(x-x')\delta(t-t') \tag{3.24}$$

3.3.2 the Gaussian model

If we neglect the S^4 term in $\mathcal{H}$ (which is valid outside the critical region or for $d > 4$), we end up with *linear* equations :

For model A :

$$\partial_t S = -D(-\nabla^2 S + r_0 S) + \zeta \tag{3.25}$$

Taking Fourier transforms :

$$\dot{S}_k = -D(k^2 + \xi_0^{-2})S_k + \zeta_k \tag{3.26}$$

where we have identified the static correlation length ξ_0.

Each mode decays independently with $\langle S_k \rangle \sim e^{-t/\tau_k}$, where

$$\tau_k = \frac{1}{D_0}(k^2 + \xi_0^{-2}) \tag{3.27}$$

Note that $\tau_0 \propto \xi_0^2 \to \infty$ at $T = (T_c)_{MF}$: this is the *critical slowing down.*

We can work out the response function in this approximation : we add a field $h(t)$ to $\mathcal{H}$:

$$\langle \dot{S}_k \rangle = -D(k^2 + \xi_0^{-2})\langle S(k) \rangle - Dh_k(t) \tag{3.28}$$

and then Fourier transform with respect to time as well :

$$\langle S_k(\omega) \rangle = h_k(\omega) \cdot G_0(\omega, k) \tag{3.29}$$

where

$$G_0(\omega, k) = \frac{1}{\frac{-i\omega}{D} + k^2 + \xi_0^{-2}} \tag{3.30}$$

In the static limit $\omega = 0$ this reproduces the Ornstein-Zernicke form.

Similarly, solving in the presence of noise but with $h = 0$ we find :

$$S_k(\omega) = \frac{\zeta_k(\omega)}{-i\omega + D(k^2 + \xi_0^{-2})} \tag{3.31}$$

and

$$\langle \zeta_k(\omega) \zeta_{k'}(\omega') \rangle = 2D\delta(\omega + \omega')\delta^{(d)}(k + k') \tag{3.32}$$

Hence,

$$C_0(\omega, k) = \frac{2D}{\omega^2 + D^2(k^2 + \xi_0^{-2})^2} = \frac{2}{\omega}\mathrm{Im}G_0 \tag{3.33}$$

so FDT is satisfied.

For <u>model B</u>, on the other hand, $D \longrightarrow D'k^2$, so

$$G_0(\omega, k) = \frac{k^2}{\frac{-i\omega}{D'} + k^2(k^2 + \xi_0^{-2})} \tag{3.34}$$

and

$$\tau_k \propto \frac{1}{k^2(k^2 + \xi_0^{-2})} \tag{3.35}$$

Modes with $k \sim \xi_0^{-1}$ decay therefore with $\tau \sim \xi_0^4$.

3.4 Response function formalism

There is a way of writing the Langevin equations in $d+1$ dimensions so they look rather like *equilibrium* models in $d+1$ space dimensions, which is very suggestive.
For example, for model A :

$$\partial_t S = -D\frac{\delta \mathcal{H}}{\delta S} + \zeta \tag{3.36}$$

We are interested in solving this equation for $S(x,t)$ for a given $\zeta(x,t)$ and then computing averages of quantities like $S(x_1,t_1)S(x_2,t_2)$ over the noise ζ. We can do this by writing

$$\left\langle \int \mathcal{D}S \, S(x_1,t_1)S(x_2,t_2)\delta[S(x,t) = \text{solution}] \right\rangle_{noise} \tag{3.37}$$

with

$$\delta[\text{equation}] = \delta\left[\dot{S} + D\frac{\delta \mathcal{H}}{\delta S} - \zeta\right] \times \text{Jacobian} \tag{3.38}$$

A word about this Jacobian. One way is to write it as

$$\det\left[\partial_t + D\frac{\delta^2 \mathcal{H}}{\delta S \delta S}\right] \tag{3.39}$$

and write this as a Grassmann integral over anticommuting fields $\psi(x,t)$, $\bar{\psi}(x,t)$:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, \exp\left(-\int \bar{\psi}\left[\partial_t + D\frac{\delta^2 \mathcal{H}}{\delta S \delta S}\right]\psi \, dt d^d x\right) \tag{3.40}$$

But in fact this is unnecessary if we regularise properly : if we interpret $\partial_t S$ as a *forward* difference operator, then

$$S(t+\delta t) \approx S(t) + \delta t\left[-D\frac{\delta \mathcal{H}}{\delta S(x,t)} + \zeta(x,t)\right] \tag{3.41}$$

and it is easy to see that $J = 1$. But note that this choice *will* have consequences later. We now write

$$\delta\left[\dot{S} + D\frac{\delta \mathcal{H}}{\delta S} - \zeta\right] = \int \mathcal{D}\tilde{S} \, \exp\left(-\int \left[d^d x dt \, \tilde{S}\left(\dot{S} + D\frac{\delta \mathcal{H}}{\delta S} - \zeta\right)\right]\right) \tag{3.42}$$

(Note that $\tilde{S}$ should strictly be integrated along the imaginary axis — in practice since we almost always do perturbation theory, this is not important).

Finally, we can average over ζ : at each point in space-time,

$$\langle e^{-\tilde{S}\zeta}\rangle = e^{(1/2)\langle\zeta\zeta\rangle\tilde{S}^2} = e^{D\tilde{S}^2} \tag{3.43}$$

The result is that correlation functions like $\langle S(x_1,t_1)S(x_2,t_2)\rangle$ may be evaluated as function integrals with a "weight"

$$\int \mathcal{D}\tilde{S}\mathcal{D}S \, \exp\left(-\int\left[d^d x dt \, \tilde{S}\left(\dot{S} + D\frac{\delta\mathcal{H}}{\delta S} - D\tilde{S}^2\right)\right]\right) \tag{3.44}$$

$\tilde{S}$ is called the *response field*. This is because its correlators give response functions. If we add a source term $+hS$ to $\mathcal{H}$, this is the same as adding $-Dh\tilde{S}$ to the "action". So

$$\underbrace{\frac{\delta\langle S(x',t')\rangle}{\delta h(x,t)}}_{=G(x-x',t'-t)} = D\langle S(x',t')\tilde{S}(x,t)\rangle \tag{3.45}$$

We can easily show FDT from this :
Add a source $+hS$ as above. The terms involving $\tilde{S}$ are : $-Dh\tilde{S}$, $-D\tilde{S}^2$. We can shift $\tilde{S} \to \tilde{S} - h/2$ to get rid of this linear term, but this induces a term $-\frac{h}{2}\left(\dot{S} + D\frac{\delta\mathcal{H}}{\delta S}\right)$
Hence

$$\begin{aligned} G(x'-x,t'-t) &= -\frac{1}{2}\left\langle S(x',t')\left[\dot{S}(x,t) + D\frac{\delta\mathcal{H}}{\delta S(x,t)}\right]\right\rangle \\ &= \underbrace{\frac{1}{2}\dot{C}(x'-x,t'-t)}_{odd} - \underbrace{\frac{D}{2}\left\langle S(x',t')\frac{\delta\mathcal{H}}{\delta S(x,t)}\right\rangle}_{even} \end{aligned} \tag{3.46}$$

But we know that $G = 0$ for $t'-t<0$, and the last term is an *even* function of t' and t. So it must be that

$$G(x'-x,t'-t) = \dot{C}(x'-x,t'-t) \qquad \text{for } t'-t>0 \tag{3.47}$$

which is FDT.

3.5 Dynamic scaling

The simple examples we have looked at so far exhibit simple *dynamic scaling* close to a critical point where $\xi^{-1} = 0$.
For model A and the Glauber model, we found that typical time scales for

the relaxation of fluctuations of the linear size of ξ behave like $\tau \propto \xi^z$ with $z = 2$.
For model B, we found $z = 4$.

Problem : Starting from a microscopic master equation for the 1d Ising model which locally conserves the magnetisation, argue that $z = 3$ in this case.

This may be generalised to hypothesise dynamic scaling forms for dynamic correlations functions which generalise the static ones.
For example

$$\tilde{G}(k, \omega, \xi) = \xi^{2-\eta}\Phi(\xi k, \xi^z \omega) \tag{3.48}$$

(for $\omega = 0$, this is the static correlation function).
Similarly,

$$\tilde{C}(k, \omega, \xi) = \xi^{2-\eta+z}\Psi(\xi k, \xi^z \omega) \tag{3.49}$$

As we shall see, these scaling forms emerge from an RG analysis, with, however, in general non-trivial values for η, z etc.

Chapter 4

Perturbation theory for Model A

Let us start with the model A equation :

$$\begin{aligned}\dot{S} &= -D\frac{\delta \mathcal{H}}{\delta S} + \zeta &(4.1)\\ &= -D\left(-\nabla^2 S + r_0 S + u_o S^3\right) + \zeta(x,t) &(4.2)\end{aligned}$$

where $\langle \zeta(x,t)\zeta(x',t')\rangle = 2D\delta^{(d)}(x-x')\delta(t-t')$

We can set up a perturbative solution in u_0 by writing it as an integral equation :

$$\dot{S} - D\nabla^2 S + Dr_0 S = -Du_0 S^3 + \zeta \tag{4.3}$$

or

$$\left(\frac{1}{D}\frac{\partial}{\partial t} - \nabla^2 + r_0\right) S = -u_0 S^3 + \frac{1}{D}\zeta \tag{4.4}$$

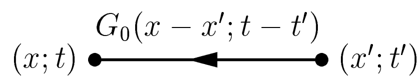
which yields

$$S(x,t) = \int d^d x dt'\, G_0(x-x', t-t')\left[-u_0 S^3(x',t') + \frac{\zeta(x',t')}{D}\right] \tag{4.5}$$

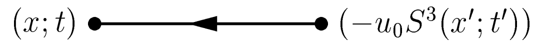
Note that G_0 is just the bare response function, with Fourier transform

$$\frac{1}{\frac{-i\omega}{D} + k^2 + r_0} \tag{4.6}$$

We can introduce a diagrammatic notation with time running from *right to left :*



Thus $S(x,t) =$



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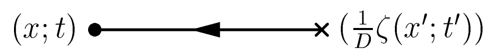
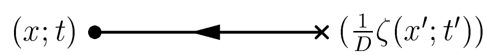


We can now iterate this equation for S :

$S(x,t) =$



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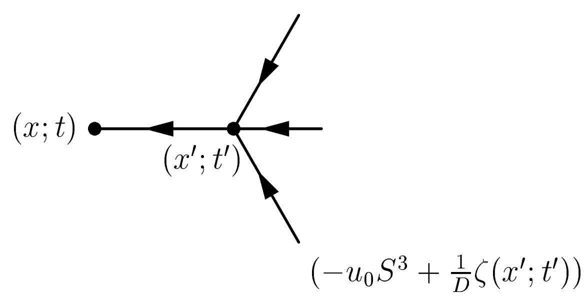

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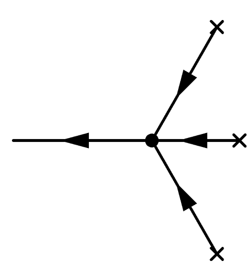

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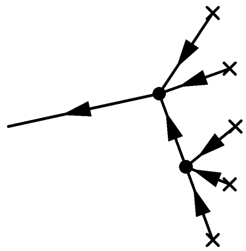
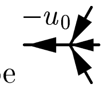


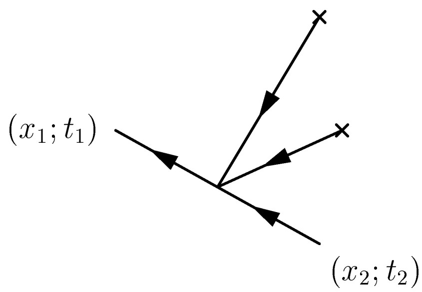
$+$ $\ldots$ where we integrate over the (x, t) of each vertex of type 

(subject to the condition that they are time-ordered), and where each  means $+\zeta/D$.

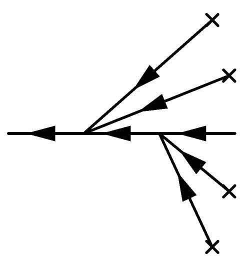
To get the *response function* we just lop off one of these x :

$G(x_1, t_1, x_2, t_2) =$

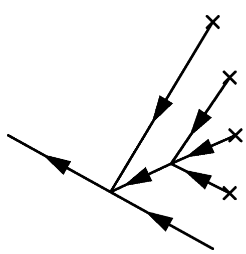
$+ \quad 3\times$



$+ \quad 3\times$



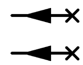
$+ \quad 2\times$

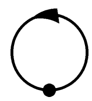


Note that, since so far all we are doing is solving a partial differential equation, the diagrams are simply trees. All counting factors are 1.

Loops come in when we average over ζ, using

$$\langle \zeta(x_1, t_1)\zeta(x_2, t_2)\rangle = 2D\delta^{(d)}(x_1 - x_2)\delta(t_1 - t_2) \tag{4.7}$$

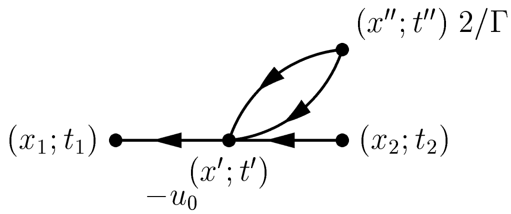
We are then supposed to tie the ends  together, in pairs, in all possible ways, with a factor $\frac{1}{D} \cdot \frac{1}{D} \cdot 2D = \frac{2}{D}$.
Note also that we never get



with our choice of regularisation of ∂_t, since the propagator  always goes forward in time.

Problem : Show that $\langle \tilde{S}\tilde{S}\rangle = 0$

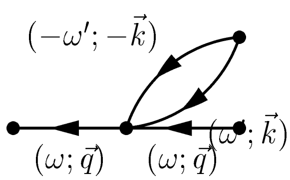
Thus, to $O(u_0^1)$, we have



which represents the term

$$= \frac{-6u_0}{D} \int_{t_1 > t' > t_2,\, t' > t''} d^dx'dt'd^dx''dt'' \times$$
$$\times\, G_0(x_1 - x', t_1 - t')G_0(x' - x_2, t' - t_2)G_0(x' - x'', t' - t'')^2 \tag{4.8}$$

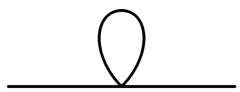
As with all Feynman diagrams, this is simpler in Fourier space :



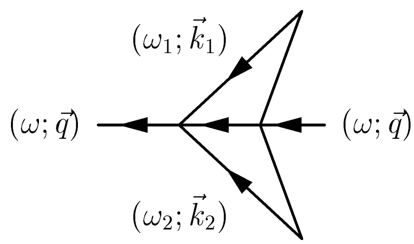
standing for

$$
\begin{aligned}
&= \frac{-6u_0}{D}\left(\frac{1}{-i\omega/D+q^2+r_0}\right)^2 \times \\
&\quad \times \int \frac{d\omega'}{2\pi}\frac{d^dk}{(2\pi)^d}\, \underbrace{\frac{1}{-i\omega'/D+k^2+r_0}\frac{1}{+i\omega'/D+k^2+r_0}}_{D\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{2(k^2+r_0)}}
\end{aligned}
\tag{4.9}
$$

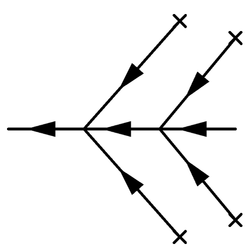
Note that at $\omega = 0$, we get the static correlation function at 1-loop :



A more interesting diagram is :



which comes from sewing in two ways the following diagram :



The overall factor is $3^2 \cdot 2 \cdot (\frac{2}{D})^2 \cdot (-u_0)^2$ and the integral is :

$$
\begin{aligned}
&\frac{1}{(-i\omega/D+q^2+r_0)^2}\int \frac{d\omega_1}{2\pi}\frac{d^dk_1}{(2\pi)^d}\frac{d\omega_2}{2\pi}\frac{d^dk_2}{(2\pi)^d} \\
&\quad\times \frac{1}{-i\omega_1/D+k_1^2+r_0}\frac{1}{+i\omega_1/D+k_1^2+r_0} \\
&\quad\times \frac{1}{-i\omega_2/D+k_2^2+r_0}\frac{1}{\frac{-i(\omega-\omega_1-\omega_2)}{D}+(q-k_1-k_2)^2+r_0}
\end{aligned}
\tag{4.10}
$$

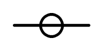
=

$$\frac{1}{(-i\omega/D+q^2+r_0)^2} D^2 \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d}$$
$$\times \frac{1}{2(k_1^2+r_0)} \frac{1}{(2k_2^2+r_0)}$$
$$\times \frac{1}{-i\omega/D+k_1^2+k_2^2+(q-k_1-k_2)^2+3r_0} \tag{4.11}$$

[Note that we can do $\omega-$ integrals automatically by looking at intermediate states in "old-fashioned" perturbation theory.

At $\omega = 0$ we can symmetrise $(k_3 = q - k_1 - k_2)$

$$\frac{1}{k_1^2+r_0} \frac{1}{k_2^2+r_0} \frac{1}{k_1^2+k_2^2+k_3^2+3r_0}$$
$$\longrightarrow \frac{1}{3} \frac{1}{k_1^2+k_2^2+k_3^2+3r_0} \left[\frac{1}{k_1^2+r_0} \frac{1}{k_2^2+r_0} + \text{perms.} \right]$$
$$= \frac{1}{3} \frac{1}{k_1^2+r_0} \frac{1}{k_2^2+r_0} \frac{1}{k_3^2+r_0} \tag{4.12}$$

and so recover the usual 2-loop diagram in the statics : 
Note that the factors of D all cancel in the static limit, as they should.

4.1 Diagrammatic expansion via the response function formalism

These diagram may also be read off from the response formalism : Rescaling $S \to \tilde{S}/D$ (so that $\langle S\tilde{S}\rangle$ is the response function) the action is :

$$S = \int dt d^dx \left[\tilde{S} \left(\frac{\dot{S}}{D} - \nabla^2 S + r_0 S + u_o S^3 \right) - \frac{1}{D} \tilde{S}^2 \right] \tag{4.13}$$

If we look at the gaussian terms $\tilde{S} \ldots S$ we see the propagator

$$\frac{1}{-i\omega/D+k^2+r_0}$$

[Note that we could include $\frac{1}{D}\tilde{S}^2$ as part of the gaussian term. This leads to a matrix

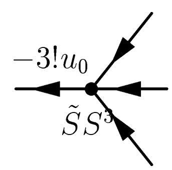
$$\begin{pmatrix} & \frac{-i\omega}{D} + k^2 + r_0 \\ \frac{+i\omega}{D} + k^2 + r_0 & -\frac{2}{D} \end{pmatrix} \tag{4.14}$$

to be inverted. Its elements are

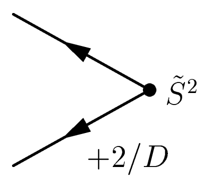
$$\begin{pmatrix} & \langle S\tilde{S}\rangle_0 \\ \langle S\tilde{S}\rangle_0^* & \langle SS\rangle_0 \end{pmatrix} \tag{4.15}$$

where the lower right element is nothing but the bare correlation function. This introduces two kinds of propagators which are indeed related by FDT. In fact, since to any finite order in u_0 we only get a finite number of $\tilde{S}^2$ vertices, it is easier to think of $\tilde{S}^2$ as part of the "interaction".]

We have vertices



and



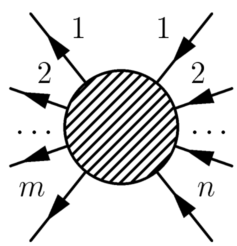
Chapter 5

RG calculations for Model A and Model B

5.1 Renormalisation of dynamic field theory (model A)

The theory as stands is regularised at large (ω, k) by lattice or other short-time effects. We can examine how the regularisation enters by power-counting. For simplicity, let us choose a cut-off in $|k| < \Lambda$: later, for elegance, we'll use dimensional regularisation.

Define $G^{(m,n)}\left((\omega_1, k_1)\cdots(\omega_m, k_m);(\omega'_1, k'_1)\cdots(\omega'_n, k'_n)\right)$ to be the connected response function with n ingoing and m outgoing lines :



From this define the truncated 1-particle irreducible vertex functions $\Gamma^{(m,n)}$. As in the static theory, only a finite number of these contain *primitive divergences* near $d = 4$. These are

$$\left[\Gamma^{(1,1)}\right] \propto k^2 \quad , \quad \left[\Gamma^{(1,3)}\right] \propto k^0 \tag{5.1}$$

Hence in $d = 4$, $\Gamma^{(1,1)}(\omega, \vec{k})$ is quadratically divergent $\propto \Lambda^2$ and $\partial_{k^2}\Gamma^{(1,1)}$, $\partial_\omega \Gamma^{(1,1)}$, $\Gamma^{(1,3)}$ are log-divergent. Apart from $\partial_\omega \Gamma^{(1,1)}$ these are just the same divergences met in the static theory. They are removed by mass, field and coupling constant renormalisation.

Working for simplicity at the critical point, we may assume that mass renormalisation has always been done, and $\Gamma^{(1,1)}(\omega = \vec{k} = 0) = 0$.

As usual, we define

$$\begin{aligned} S_R &= Z_S^{-1/2} S \\ \tilde{S}_R &= Z_S^{-1/2} \tilde{S} \end{aligned} \tag{5.2}$$

so

$$\Gamma_R^{(1,1)} = Z_S \Gamma^{(1,1)} \tag{5.3}$$

and in general

$$\Gamma_R^{(m,n)} = Z_S^{\frac{m+n}{2}} \Gamma^{(m,n)} \tag{5.4}$$

where

$$\left(\partial_{k^2} \Gamma_R^{(1,1)}\right)_{\omega=0,k=\mu} = 1 \tag{5.5}$$

$\Big[$ this is inspired by $\Gamma^{(1,1)} = -i\omega/D + k^2$ in free theory $\Big]$

In the same way we define

$$u_R = -\left(\Gamma_R^{(1,3)}\right)_{\omega_i=0,k_i\propto\mu} \tag{5.6}$$

This suggest that we therefore define

$$\frac{1}{D_R} = \left(+\frac{\partial}{\partial(-i\omega)}\Gamma_R^{(1,1)}\right)_{\omega=0,k=\mu} \equiv \frac{1}{Z_D D_0} \tag{5.7}$$

The statement of *renormalisability* of the dynamic theory is that all response functions $\Gamma_R^{(m,n)}$ are finite as $\Lambda \to \infty$ when expressed in terms of u_R and D_R.

Let us focus on $\Gamma_R^{(1,1)} = Z_S \Gamma^{(1,1)}$

$$\Gamma_R^{(1,1)}(\omega, k, D_R, u_R, \mu) = Z_S(u_R, \mu, \Lambda)\Gamma^{(1,1)}(\omega, k, D_0, u_0, \Lambda) \tag{5.8}$$

Since $\Gamma^{(1,1)}$ does not depend on μ, we can write

$$\left(\mu\frac{\partial}{\partial\mu}\right)_{u_0,\Lambda,D_0} Z_S^{-1}\Gamma_R^{(1,1)} = 0 \tag{5.9}$$

We define

$$\underbrace{g_R}_{\text{dimensionless}} = u_R \mu^{-\epsilon} \tag{5.10}$$

and can rewrite it

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} + Z_S\mu\frac{\partial}{\partial\mu}(Z_S^{-1}) + \mu\frac{\partial D_R}{\partial\mu}\frac{\partial}{\partial D_R}\right]\Gamma_R^{(1,1)} = 0 \tag{5.11}$$

where as usual

$$\beta(g_R) = \left(\mu\frac{\partial}{\partial\mu}g_R\right)_{u_0,\Lambda} \tag{5.12}$$

We define now

$$\begin{aligned}
\gamma_s(g_R) &\equiv \left(+\frac{1}{Z_S}\mu\frac{\partial}{\partial\mu}Z_S\right)_{u_0,\Lambda} \\
\gamma_D(g_R) &\equiv \left(+\frac{1}{D_R}\mu\frac{\partial}{\partial\mu}D_R\right)_{u_0,D_0,\Lambda} \\
&= \frac{1}{Z_D}\mu\frac{\partial}{\partial\mu}Z_D
\end{aligned} \tag{5.13}$$

For simplicity suppose $g_R = g^* = \mathcal{O}(\epsilon)$ and $\beta(g^*) = 0$.

We then have

$$\left[\mu\frac{\partial}{\partial\mu} - \gamma_s^* + \gamma_D^* D_R\frac{\partial}{\partial D_R}\right]\Gamma_R^{(1,1)} = 0 \tag{5.14}$$

Now we have to use a version of dimensional analysis :

$$\Gamma_R^{(1,1)}(\omega, k, D_R, \mu) = \mu^2\Phi\left[\frac{k}{\mu}, \frac{\omega}{D_R k^2}\right] \tag{5.15}$$

From this we see that $D_R\partial_R = -\omega\partial_\omega$ and $\mu\partial_\mu + k\partial_k = 2 - 2\omega\partial_\omega$, so

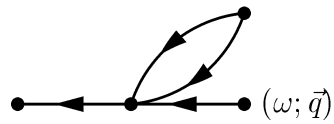
$$\left[-k\frac{\partial}{\partial k} + 2 - \gamma_s^* - (2+\gamma_D^*)\omega\frac{\partial}{\partial\omega}\right]\Gamma^{(1,1)}(\omega, k) = 0 \tag{5.16}$$

$$\Gamma^{(1,1)}(\omega, k) = k^{2-\gamma_s^*}\Phi\left[\frac{\omega}{k^{2+\gamma_D^*}}\right] \tag{5.17}$$

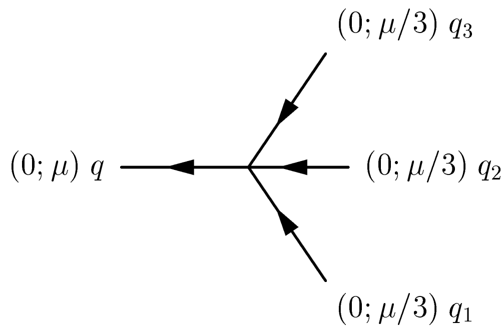
which is <u>dynamic scaling</u>, with $\eta = \gamma_s^*$, $z = 2 + \gamma_D^*$.

5.1.1 Lowest order calculation

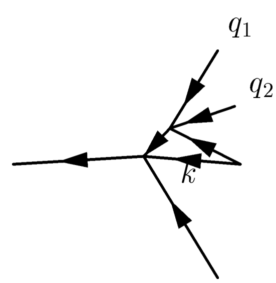
Notice there are no 1-loop corrections to Z_s or Z_Γ since the only diagram is



which, after truncating the external lines, is independent of ω and q.
To one loop, then, we have only the renormalisation of u_0



$+$ $\qquad -u_0$



$$(-u_0)^2 \times 3 \times 3 \times 2 \times (2/\Gamma)$$

The one-loop diagram is therefore

$$(-u_0)^2 \frac{3^2 \cdot 2^2}{D_0} \underbrace{D_0}_{\text{from} \int d\omega} \cdot \int \frac{d^d k}{(2\pi)^d} \frac{1}{-i0 + k^2 + (q_1 + q_2 - k)^2} \frac{1}{-i0 + k^2 + k^2} \tag{5.18}$$

This is log-divergent in $d = 4$ as expected : in $4 - \epsilon$ dimensions the integral gives

$$\frac{1}{4} \frac{2\pi^2}{(2\pi)^4} \mu^{-\epsilon} \left[\frac{1}{\epsilon} + \mathcal{O}(1) \right] \tag{5.19}$$

Thus

$$u_R = u_0 - 3^2 u_0^2 \mu^{-\epsilon} \underbrace{\frac{2\pi^2}{(2\pi)^4}}_{\equiv K_4} \frac{1}{\epsilon} + \mathcal{O}(u_0^3) \tag{5.20}$$

$$g_R = u_0 \mu^{-\epsilon} \left(1 - 3^2 \frac{K_4}{\epsilon} u_0 \mu^{-\epsilon} + \cdots\right) \tag{5.21}$$

$$\begin{aligned} \beta(g_R) \equiv \left(\mu \frac{\partial g_R}{\partial \mu}\right)_{u_0} &= -\epsilon g_R + u_0 \mu^{-\epsilon} \cdot 3^2 K_4 u_0 \mu^{-\epsilon} + \cdots \\ &= -\epsilon g_R + 3^2 K_4 g_R^2 + \cdots \end{aligned} \tag{5.22}$$

$$g_R^* = \frac{\epsilon}{3^2 K_4} + \mathcal{O}(\epsilon^2) \tag{5.23}$$

Problem : Check this is the same as in static theory. NB factor of 6 in the definition of u_0.

Now let's calculate Z_D :

$$Z_D^{-1} = D_0 \frac{\partial}{\partial(-i\omega)} \Gamma_R^{(1,1)} = Z_S D_0 \frac{\partial}{\partial(-i\omega)} \Gamma^{(1,1)} \tag{5.24}$$

$$\Gamma^{(1,1)} = \frac{-i\omega}{D_0} + q^2 \underbrace{-}_{NB}$$  $$+ \cdots$$

$$\begin{aligned} &= \frac{-i\omega}{D_0} + q^2 - 3^2 \cdot 2 \cdot (-u_0)^2 \int dk_1 dk_2 \frac{1}{k_1^2} \frac{1}{k_2^2} \cdot \\ &\quad \cdot \frac{1}{-i\omega/D_0 + k_1^2 + k_2^2 + (q - k_1 - k_2)^2} + \cdots \end{aligned} \tag{5.25}$$

so that

$$Z_D^{-1} = Z_s \left[1 + 3^2 \cdot 2 \cdot u_0^2 \int dk_1 dk_2 \, \frac{1}{k_1^2} \frac{1}{k_2^2} \, \frac{1}{[k_1^2 + k_2^2 + (q - k_1 - k_2)^2]^2} + \cdots\right] \tag{5.26}$$

at $q^2 = \mu^2$.

Problem : Show that this integral gives us $\mu^{-2\epsilon} K_4^2 \frac{A}{\epsilon}$

We then have

$$\gamma_D = -\gamma_s + 3^2 \cdot 2 \cdot u_0^2 \mu^{-2\epsilon} K_4^2 \cdot 2A \tag{5.27}$$

and since

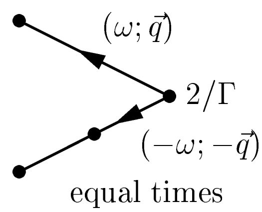
$$\gamma_s^* = \eta = \frac{\epsilon^2}{54} + \mathcal{O}(\epsilon^3) \tag{5.28}$$

this gives finally

$$z = 2 + 0.0135\epsilon^2 + \cdots \tag{5.29}$$

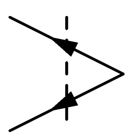
Note 1 : We chose $Z_{\tilde{s}} = Z_s$ but in principle we could shuffle these factors around : e.g. choose $Z_{\tilde{s}} = 1$. In other theories, we may well choose to do this.

Note 2 : We can also get the renormalisation of Γ by looking at the correlation function $\langle SS \rangle$, which to lowest order is given by

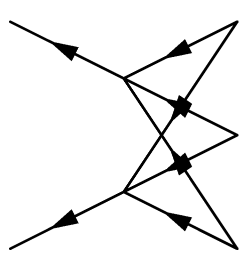


which gives the integral

$$\begin{aligned} &= \frac{2}{D_0} \int \frac{d\omega}{2\pi i} \frac{1}{-i\omega/D_0 + q^2} \frac{1}{+i\omega/D_0 + q^2} \\ &= \frac{1}{q^2} \quad \text{as expected} \end{aligned} \tag{5.30}$$

[or by cutting  $\rightarrow 2 \cdot \frac{1}{-i0+2q^2}$]

The 2-loop correction to this is :



Problem : Check that this gives the symmetrised version of the previous integral.

5.2 Renormalisation of model B

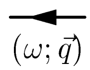
Model B corresponds to a conserved order parameter, as appropriate to a binary fluid (ignoring however hydrodynamic effects !)
In this case,

$$\dot{S} = D'\nabla^2\left(-\nabla^2 S + r_0 S + u_0 S^3\right) + \zeta \tag{5.31}$$

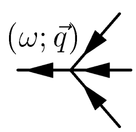
where

$$\langle \zeta(x,t)\zeta(x',t')\rangle = -2D'\nabla^2\delta^{(d)}(x-x')\delta(t-t') \tag{5.32}$$

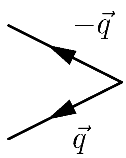
Thus the differences from model A are :

1. The bare propagator is 

$= \left(\frac{-i\omega}{D_0} + q^2(q^2+r_0)\right)^{-1}$

2. The vertex 

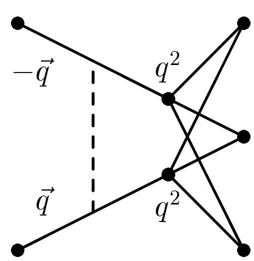
$= -u_0 q^2$

3. The noise 

$= \frac{2}{D_0'}q^2$

Dimensional analysis goes through in the same way for the $\Gamma^{(m,n)}$, so in principle, $\Gamma^{(1,1)}$, $\frac{\partial\Gamma^{(1,1)}}{\partial q^2}$, $\frac{\partial\Gamma^{(1,1)}}{\partial(-i\omega)}$ and $\Gamma^{(1,3)}$ show primitive divergences.

But ! : The factor q^2 in the vertex makes a big difference. In fact, if we look at the renormalisation of $\langle SS\rangle$ again, we have



$$\langle SS\rangle = \frac{1}{q^2(q^2+r_0)}\left[q^2 + (q^2)^2 u_0^2 \times \text{some integral} + \cdots\right] \tag{5.33}$$

Since u_0 is dimensionless at $d = 4$, this integral cannot be divergent !

Problem : check that this is indeed true.

As a result, then, $Z_D^{-1} = Z_S$ and $\gamma_D = -\gamma_S$. So,

$$z = 2 - \eta \qquad \text{to all orders in } \epsilon \tag{5.34}$$

Check : for $d = 1$, $\eta = -1$ (why ?) and $z = 3 \quad \Rightarrow$ OK.

Chapter 6

More realistic models

It turns out to be almost impossible to find real physical systems accurately described by models A or B. This is because in real systems, there are other *slow modes* which interact with the order parameter modes. They do not affect the statics, but may have a dramatic effect on the dynamics.

Two examples :

1. Effect of slow heat conduction

Since equilibration occurs via contact with a heat bath (= phonons), and they diffuse as well as the spin modes, we should include these phonons. The effective hamiltonian has the form

$$\mathcal{H} = \int d^d x \left(\frac{1}{2}(\nabla S)^2 + \frac{1}{2} r_0 S^2 + \frac{1}{4} u_0 S^4 + \frac{1}{2}\rho^2 + \frac{1}{2} g \rho S^2 \right) \tag{6.1}$$

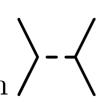
where

- ρ is the energy density of phonons, in units where heat capacity = 1.
- g is the coupling between the phonons and the spin degrees of freedom.

We can ignore terms like $(\nabla \rho)^2$ since ρ is not critical itself.

For the statics, ρ makes no difference, since

$$\int d\rho \, \exp\left(-\frac{1}{2}\rho^2 + \frac{1}{2} g \rho S^2\right) \quad \propto \quad \exp\left(g^2 S^4 / 8\right) \tag{6.2}$$

which simply shifts u_0 (This corresponds to the diagram )

For the dynamics, we have

$$\dot{S} = -D\frac{\delta\mathcal{H}}{\delta S} + \zeta \tag{6.3}$$

$$\begin{aligned} \dot{\rho} &= +D'\nabla^2\frac{\delta\mathcal{H}}{\delta\rho} + \eta \qquad (\text{since}\,\rho\,\text{is conserved}) \\ &= D'\nabla^2\left(\rho + \frac{1}{2}gS^2\right) + \eta \end{aligned} \tag{6.4}$$

showing how the energy of the spin degrees of freedom drives heat conduction.

In the same way,

$$\dot{S} = -D\left(\nabla^2 S + r_0 S + u_0 S^3 + g\rho S\right) + \zeta \tag{6.5}$$

showing that ρ acts like a local variation of T_c.

From these we also see that $[g\rho] = [u_0][S^2] = [u_0][\rho]/[g]$, so $[g^2] = [u_0]$ and g is also dimensionless in $d = 4$.

Problem : Draw diagrams which renormalise g and u_0 to 1-loop. Is D renormalised now at 1-loop ?

2. Isotropic ferromagnet

As well as relaxational modes, there may also be organised motion in the system which corresponds to "real" dynamics.
For example in a Heisenberg ferromagnet, the local magnetization $\vec{S}(x,t)$ will precess in the local field $\vec{B}$ according to

$$\dot{\vec{S}} \propto \vec{S} \times \vec{B} \tag{6.6}$$

Such precession may be deduced from the quantum equations of motion and will not disappear on coarse-graining. The local field $\vec{B}$ depends on $\vec{S}(x',t)$ for x' near x, hence to lowest order in derivatives, in may be written $\vec{B} \propto \vec{S} + \text{const}\nabla^2\vec{S} + \cdots$. Since $\vec{S} \times \vec{S} = 0$, we find a term

$$\dot{\vec{S}} = \lambda\vec{S} \times \nabla^2\vec{S} \;+\; \text{model B terms} \;+\; \text{noise} \tag{6.7}$$

Note that such so-called *reversible* terms do not spoil FDT or the approach to the equilibrium distribution :

$$\langle \delta(S - S_0) \rangle = P[S_0] \propto e^{-\mathcal{H}[S_0]} \tag{6.8}$$

$$\langle \delta(S - S_0) \rangle = Z^{-1} \int \mathcal{D}S \, \delta(S - S_0) e^{-\mathcal{H}(S)} \tag{6.9}$$

$$\begin{aligned} \frac{d}{dt} P[S_0(t)] &= -Z^{-1} \int \mathcal{D}S \sum_x \delta'(S - \dot{S}_0) e^{-\mathcal{H}(S)} \\ &= \int d^d x \, \dot{S}_0 \left(\frac{\delta}{\delta S} e^{-\mathcal{H}(S)} \right)_{S=S_0} \\ &= -\int d^d x \left\langle \dot{S}_0 \frac{\delta \mathcal{H}}{\delta S_0} \right\rangle \end{aligned} \tag{6.10}$$

For model A, this is

$$\left\langle \left(-D \frac{\delta \mathcal{H}}{\delta S_0} + \zeta \right) \frac{\delta \mathcal{H}}{\delta S_0} \right\rangle = 0 \tag{6.11}$$

If we add the term $\vec{S} \times \nabla^2 \vec{S}$, we have

$$\int d^d x \left\langle (\vec{S} \times \nabla^2 \vec{S}) \frac{\delta \mathcal{H}}{\delta S} \right\rangle \tag{6.12}$$

which vanishes by symmetry.

By looking at the model B terms $-D' \nabla^4 S - D' \nabla^2 r_0 S + \cdots$, we see that

$$[\lambda / D'] = k^2 [S]^{-1} = k^{3-d/2} \tag{6.13}$$

so that λ / D' is relevant when $d < 6$!

Scaling suggests that since $\dot{S} = \lambda S \times \nabla^2 S$:

$$[\omega] = [k^2][S] = [k^2] k^{\frac{d-2+\eta}{2}} \tag{6.14}$$

$$z = \frac{d + 2 - \eta}{2} \tag{6.15}$$

which is true to all orders due to a Ward identity.

Part II

Non-equilibrium phase transitions

Chapter 7

Introduction

In the case of critical dynamics near equilibrium, we were guided by the principles of detailed balance, Einstein relations, FDT, etc. to a form of the Langevin equation which was largely dictated. But for systems driven (or relaxing) far from equilibrium, this is no longer valid.

For simplicity, we shall consider only stochastic *particle systems* (e.g. reaction-diffusion models, simple fluids, etc.)

As a very simple example, consider the reaction-diffusion model where a single species of particles A do random walks on a lattice and, whenever they meet on the same site, undergo the reaction $A + A \to \emptyset$(inert) at rate λ. (we allow multiple occupation : if the mean density is small, this is unlikely anyway).
For this process, we might write down the *rate equation* for the mean density $n(x, t)$:

$$\frac{\partial n}{\partial t} = D\nabla^2 n - 2\lambda n^{(2)} \tag{7.1}$$

where

- D is the diffusion coefficient of the A particles on the lattice.
- $n^{(2)}$ is the probability of finding 2 particles on the same site.

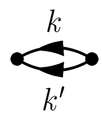
In the spirit of the mean-field approximation, we might write

$$n^{(2)} \approx n^2 \tag{7.2}$$

in which case equation (7.1) is easy to solve :

$$n(t) = \frac{n_0}{1 + 2\lambda n_0 t} \tag{7.3}$$

in the homogeneous case. Note that as $t \to \infty$, $n(t) \sim (\lambda t)^{-1}$ independently of n_0 but not of λ.
Approximation (7.2) is valid as long as the fluctuations $n^{(2)} - \langle n \rangle^2$ are small. These are caused by particles having been in the same region of space at some previous time, and are given to lowest order by the diagram

$\int_{t>t'} d^d x' dt'$ $(x;t)$  $(x';t')$ $\propto \int \frac{d^d k}{(2\pi)^d} \frac{1}{2Dk^2}$

For $d > 2$, this is finite (with some UV cut-off $\propto$ (lattice spacing)$^{-1}$) but for $d \leq 2$ it diverges., due to the recurrence property of simple random walks.
We might hope to account for such effects by adding a noise term, as in equilibrium problems :

$$\dot{n} = D\nabla^2 n - 2\lambda n^2 + \zeta \tag{7.4}$$

but we have no obvious way of fixing the correlations. As we shall see, (7.4), at least taken literally, is simply wrong.

Instead, we will adopt a different approach, summarized by the following flow chart :

Master eqn	$\longrightarrow$	"Second-quantised" version	$\longrightarrow$	Path integral		
				$\downarrow$		
		Analyse using field-th. RG	$\longrightarrow$	Continuum limit	$\longrightarrow$	interpretation as Langevin eqn

We will initially consider the reaction $A + A \to \emptyset$ for simplicity, and generalise later.

Chapter 8

Field-theoretic representation of the master equation

8.1 Basic principles

As stated before, the master equation has the form :

$$\frac{dP(\alpha)}{dt} = \sum_{\beta} R_{\beta\to\alpha} P(\beta) - \sum_{\beta} R_{\alpha\to\beta} P(\alpha) \tag{8.1}$$

On a lattice $\mathcal{L}$, the microstates α are the occupation numbers $\{n\} \equiv \{n_1, n_2, ...\}$ of each site. Equation (8.1) is like the Schrödinger equation for a many-body wave function in that it is

1. linear in the $P(\alpha)$
2. first-order in $\partial/\partial t$

This suggests a "second-quantised" formalism :

- Define
$$p(n_1, n_2, ...; t) = P(\{n\}, t) \tag{8.2}$$
- Introduce annihilation & creation operators $\{a_i, a_i^\dagger\}_{i\in\mathcal{L}}$ with $[a_i, a_i^\dagger] = 1$.
- Define the $|0\rangle$ state as satisfying $a_i|0\rangle = 0 \ \forall i$
- Define now $|\Psi(t)\rangle = \sum_{\{n_i\}} p(\{n\}, t) a_1^{\dagger n_1} a_2^{\dagger n_2} ...|0\rangle$

Then the master equation is completely equivalent to the Schrödinger-like equation

$$\frac{d}{dt}|\Psi(t)\rangle = -H|\Psi(t)\rangle \tag{8.3}$$

where H is an operator depending on the a's & $a^\dagger$'s only.

Let's now work towards this result with a couple of examples :

8.2 Example a : Simple hopping

Consider just 2 sites (1,2) and hopping $1 \to 2$ at rate D. The master equation is :

$$\frac{dP(n_1,n_2)}{dt} = D(n_1+1)P(n_1+1,n_2-1) - Dn_1P(n_1,n_2) \tag{8.4}$$

Notice that the actual rates are proportional to n_1, since each particle may hop independently. [We could modify this if we wanted.]

Defining $|\Psi\rangle = \sum_{n_1,n_2} a_1^{\dagger n_1} a_2^{\dagger n_2} P(n_1,n_2)|0\rangle$ we get :

$$\begin{aligned}
\frac{d|\Psi\rangle}{dt} &= D\sum_{n_1,n_2}\left[P(n_1+1,n_2-1)(n_1+1) - P(n_1,n_2)n_1\right] a_1^{\dagger n_1} a_2^{\dagger n_2}|0\rangle \\
&= D\sum_{n_1,n_2} P(n_1+1,n_2-1) a_2^\dagger a_1 a_1^{\dagger(n_1+1)} a_2^{\dagger(n_2-1)}|0\rangle - \\
&\quad D\sum_{n_1,n_2} P(n_1,n_2) a_1^\dagger a_1 a_1^{\dagger n_1} a_2^{n_2}|0\rangle \qquad (8.5)\\
&= D(a_2^\dagger a_1 - a_1^\dagger a_1)|\Psi\rangle \qquad (8.6)
\end{aligned}$$

(using $a_i \leftrightarrow \partial/\partial a_i^\dagger$). So in this case,

$$H \equiv -D(a_2^\dagger a_1 - a_1^\dagger a_1) \tag{8.7}$$

Note that as well as the obvious & intuitive hopping term $a_2^\dagger a_1$, we have a term $-a_1^\dagger a_1$ which ensures probability conservation.
If we consider also hopping $2 \to 1$ at the same rate, we have :

$$\begin{aligned}
H &= -D(a_2^\dagger a_1 - a_1^\dagger a_1 + a_1^\dagger a_2 - a_2^\dagger a_2) \\
&= D(a_2^\dagger - a_1^\dagger)(a_2 - a_1) \qquad (8.8)
\end{aligned}$$

8.3 Example b : Simple annihilation at a single site

Here the master equation is :

$$\frac{dP(n)}{dt} = \lambda(n+2)(n+1)P(n+2) - \lambda n(n-1)P(n) \tag{8.9}$$

$$|\Psi\rangle = \sum_n P(n) a^{\dagger n}|0\rangle \tag{8.10}$$

$$\begin{aligned} \frac{d|\Psi\rangle}{dt} &= \lambda \sum_n (n+2)(n+1)P(n+2)a^{\dagger n}|0\rangle - \lambda \sum_n n(n-1)P(n)a^{\dagger n}|0\rangle \\ &= \lambda \sum_n P(n+2)a^2 a^{\dagger(n+2)}|0\rangle - \lambda \sum_n P(n)a^{\dagger 2}a^2 a^{\dagger n}|0\rangle \end{aligned} \tag{8.11}$$

so

$$H = -\lambda(a^2 - a^{\dagger 2}a^2) \tag{8.12}$$

Once again, as well as the obvious term proportional to a^2, there is another diagonal term in H.
Putting these together for our lattice model, we obtain :

$$H = D\sum_{\langle ij\rangle}(a_i^\dagger - a_j^\dagger)(a_i - a_j) - \lambda\sum_i (a_i^2 - a_i^{\dagger 2}a_i^2) \tag{8.13}$$

8.4 Aspects of this formalism which differ from ordinary many-body QM :

1. No "i" in the Schrödinger equation — like "euclidean" QM.
2. H is not (necessarily) hermitian.
   Problem : Show that if the rates satisfy *detailed balance*, then H may be made symmetric & real by a similarity transformation.
3. **Most important** : expectation values of observables $A(n_1, n_2, ...)$ are NOT $\langle\Psi(t)|A|\Psi(t)\rangle$ since this would be bilinear in the $p(\{n_i\})$

Instead, we have :

$$\begin{aligned} \bar{A} &= \sum_{\{n_i\}} p(n_1, n_2, ...)A(n_1, n_2, ...) \\ &= \langle 0|e^{\sum_i a_i}A(n_1, n_2, ...)\sum_{\{n_i\}} p(n_1, n_2)a_1^{\dagger n_1}a_2^{\dagger n_2}...|0\rangle \\ &= \langle\Psi_0|A|\Psi(t)\rangle \\ &= \langle\Psi_0|Ae^{-Ht}|\Psi(0)\rangle \end{aligned} \tag{8.14}$$

where $\langle\Psi_0| = \langle 0|e^{\sum_i a_i}$.
$\Big[$ Proof : use $[e^a, a^\dagger] = \left[e^a, -\frac{\partial}{\partial a}\right] = e^a$ and $\langle 0|a^\dagger = 0.$ $\Big]$

An immediate corollary of this is a condition that H conserves probability :

$$1 = \bar{1} = \left\langle \Psi_0 | e^{-Ht} | \Psi(0) \right\rangle \tag{8.15}$$

so

$$\begin{aligned} \langle \Psi_0 | H &= 0 \\ \text{and} \qquad \langle \Psi_0 | \Psi(0) \rangle &= 1 \end{aligned} \tag{8.16}$$

Since $\langle \Psi_0 | a_i^\dagger = 1$ this is equivalent to the condition that H vanishes if we formally set each $a_i^\dagger$ to 1.
This factor of $e^{\sum_i a_i}$ may or may not be a problem, depending on the nature of $A(n_1, n_2, ...)$.
If we are interested in *exclusive* probabilities, e.g. the probability that there is exactly 1 particle at site 1 and zero particles elsewhere, then

$$A = \delta_{a_1^\dagger a_1, 1} \prod_{j \neq 1} \delta_{a_j^\dagger a_j, 0} \tag{8.17}$$

and the factor $\langle 0 | e^{\sum_i a_i}$ becomes simply $\langle 0 | a_1$.
If, however, we are interested in *inclusive* probabilities, e.g. the average number of particles at site 1 *irrespective* of other sites, we need

$$\langle 0 | e^{\sum_i a_i} a_1^\dagger a_1 e^{-Ht} | \Psi(0) \rangle = \langle 0 | e^{\sum_i a_i} a_1 e^{-Ht} | \Psi(0) \rangle \tag{8.18}$$

In this case, it is easier to commute the factor $e^{\sum_i a_i}$ through, using

$$e^a a^\dagger = (a^\dagger + 1) e^a \tag{8.19}$$

so to get

$$\langle 0 | a_1 e^{-H(\{a^\dagger + 1, a\})t} | \tilde{\Psi}(0) \rangle \tag{8.20}$$

where $H(\{a^\dagger + 1, a\})$ may be called a "shifted" hamiltonian and $|\tilde{\Psi}(0)\rangle \equiv e^{\sum_i a_i} |\Psi(0)\rangle$.
In our case, the shifted hamiltonian is

$$H = D \sum_i (a_i^\dagger - a_j^\dagger)(a_i - a_j) + \lambda \sum_i (2 a_i^\dagger a_i^2 + a_i^{\dagger 2} a_i^2) \tag{8.21}$$

For an initial state, a suitable choice is

$$\Psi(0) = e^{-n_0} e^{n_0 \sum_i a_i^\dagger} |0\rangle \tag{8.22}$$

corresponding to a *Poisson* distribution $p(n, 0) = e^{-n_0} \frac{n_0^n}{n!}$ at each site.
In this case, $|\tilde{\Psi}(0)\rangle = e^{n_0 \sum_i a_i^\dagger} |0\rangle$.

Chapter 9

Path integral representation

Once again, for simplicity, consider a single site.
We want to evaluate e^{-Ht}. We write this as a product :

$$e^{-Ht} = \lim_{\Delta t \to 0} (1 - H\Delta t)^{t/\Delta t} = \underbrace{(1 - H\Delta t) \cdot (1 - H\Delta t) \cdots}_{t/\Delta t \,\text{factors}} \tag{9.1}$$

Into each time slice, we insert a complete set of coherent states :

$$\begin{aligned} \int \frac{d\phi^* d\phi}{\pi}\, e^{-\phi^*\phi} e^{\phi a^\dagger} |0\rangle\langle 0| e^{\phi^* a} = \\ = \int \frac{d\phi^* d\phi}{\pi}\, e^{-\phi^*\phi} \sum_{m,n} \frac{\phi^m \phi^{*n}}{m!n!} a^{\dagger m} |0\rangle\langle 0| a^n \end{aligned} \tag{9.2}$$

Terms with $m \neq n$ give zero on integrating over the phase of ϕ. Letting $|\phi| \equiv \rho$, we get

$$\begin{aligned} &= \int_0^\infty 2\rho d\rho\, e^{-\rho^2} \sum_n \left[\frac{(\rho^2)^n}{n!} \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle\langle 0| \frac{a^n}{\sqrt{n!}} \right] \\ &= \sum_n \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle\langle 0| \frac{a^n}{\sqrt{n!}} = 1 \end{aligned} \tag{9.3}$$

Between each slice, we have :

$$\begin{aligned} &\langle 0| e^{\phi^*(t+\Delta t)a} (1 - \Delta t H) e^{\phi(t) a^\dagger} |0\rangle = \\ &\quad = e^{\phi^*(t+\Delta t)\phi(t)} - \Delta t \langle 0| e^{\phi^*(t)a} H e^{\phi(t)a^\dagger} |0\rangle + \mathcal{O}((\Delta t)^2) \\ &\quad = e^{\phi^*(t+\Delta t)\phi(t)} e^{-\Delta t H(\phi^*,\phi)} + \mathcal{O}((\Delta t)^2) \end{aligned} \tag{9.4}$$

where $H(\phi^*, \phi)$ is obtained by replacing $a \to \phi$, $a^\dagger \to \phi^*$.

The remaining terms are

$$\prod e^{\phi^*(t+\Delta t)\phi(t)-\phi^*(t+\Delta t)\phi(t+\Delta t)} \quad \approx \quad e^{-\int dt\, \phi^* \partial_t \phi} \tag{9.5}$$

so we get, in the limit $\Delta t \to 0$, a functional integral (generalising to $d \neq 0$)

$$\int \mathcal{D}\phi^* \mathcal{D}\phi \; e^{-\int dt d^d x \; \mathcal{L}(\phi^*, \phi)} \tag{9.6}$$

where

$$\int \mathcal{L} d^d x = \int d^d x \; \phi^* \partial_t \phi + H(\phi^*, \phi) \tag{9.7}$$

$$\mathcal{L} = \phi^* \partial_t \phi + D(\nabla \phi^*)(\nabla \phi) - \lambda(\phi^2 - \phi^{*2}\phi^2) \tag{9.8}$$

or, before taking the continuum limit,

$$\mathcal{L} = \sum_j \phi_j^* \partial_t \phi_j + D \sum_{\langle ij \rangle} (\phi_i^* - \phi_j^*)(\phi_i - \phi_j) - \lambda \sum_i (\phi_i^2 - \phi_i^{*2}\phi_i^2) \tag{9.9}$$

Note that we do not need to coarse-grain to get a field theory on the lattice.

In the same way, we can show that the following factors go over respectively into :

$$\begin{aligned} e^{\sum_i a_i} &\longrightarrow e^{\sum_j \phi_j} \\ e^{-n_0 \sum_i a_i^\dagger} &\longrightarrow e^{-n_0 \sum_j \phi_j^*} \end{aligned} \tag{9.10}$$

We can get rid of the first term by shifting

$$\phi_j^* = 1 + \tilde{\phi}_j \tag{9.11}$$

The extra term $e^{-\int dt\, \partial_t \phi_j}$ integrates up to cancel e^{ϕ_j}.

Similarly, observables like $A(n_j)$ give $A(\phi_j^* \phi_j)$ and so on.

Chapter 10

Interpretation as a Langevin equation

For simplicity, let us write the continuum form, with the shift :

$$\exp\left\{-\int dt d^d x\,[\tilde{\phi}\partial_t\phi + D(\nabla\tilde{\phi})(\nabla\phi) + 2\lambda\tilde{\phi}\phi^2 + \lambda\tilde{\phi}^2\phi^2]\right\} \tag{10.1}$$

where the $D(\nabla\tilde{\phi})(\nabla\phi)$ term can be integrated by parts to give $-D\tilde{\phi}\nabla^2\phi$ + surface term.

This looks very like the response function formalism we discussed earlier. We can undo the quadratic $\tilde{\phi}^2$ term by writing

$$\exp\left\{-\lambda\int dt d^d x\ \tilde{\phi}^2\phi^2\right\} = \int \mathcal{D}\zeta\ \exp\left\{dt d^d x\ \tilde{\phi}\zeta\right\} P([\zeta]) \tag{10.2}$$

where $P()$ is the "probability distribution" for the "noise" ζ.

The action is now linear in $\tilde{\phi}$ and we can integrate it out to obtain a Langevin equation :

$$\partial_t\phi = D\nabla^2\phi - 2\lambda\phi^2 + \zeta \tag{10.3}$$

If we neglect ζ, we recognise this as the rate equation we wrote earlier, if we interpret ϕ as the average density. In fact this is so, at this level, because

$$\overline{a^\dagger a} = \langle 0|e^a a^\dagger a|\Psi\rangle \longrightarrow \langle\phi\rangle \tag{10.4}$$

where $\langle\cdots\rangle$ denotes the average with respect to the weight e^{-S}.

But, if we are careful with the signs, we see that

$$\langle\zeta(x,t)\zeta(x',t')\rangle = -2\lambda\phi^2(x,t)\delta^d(x-x')\delta(t-t') \tag{10.5}$$

The appearance of ϕ^2 makes sense : if there are no particles, there is no noise. But the *sign* means that ζ is pure imaginary !

How can this be ? The answer is that although $\langle\phi\rangle$ is the *average* density, $\phi(x,t)$ is NOT the density. In fact,

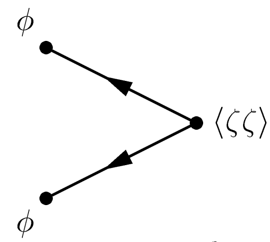
$$\overline{n^2} = \langle 0|e^a(a^\dagger a)^2|\Psi\rangle = \langle 0|e^a(a^2+a)|\Psi\rangle \longrightarrow \langle\phi+\phi^2\rangle \tag{10.6}$$

It is easy to check (Problem) that if all the higher cumulants $\langle\phi^2\rangle - \langle\phi\rangle^2$ etc. of ϕ vanish, as would be true in the absence of loops in our field theory, then the actual density n would have a *Poisson* distribution, as expected.

Another reason why ζ is imaginary can be seen by studying the equal-time density-density correlation function

$$\overline{n(x,t)n(x',t')} = \langle\phi(x,t)\phi(x',t')\rangle \qquad \text{for } x \neq x' \tag{10.7}$$

The lowest order diagram is



which is negative. So particles are anti-correlated. This makes sense : there is a *deficit* of particles in the neighbourhood of a given one, since any particles nearby have been "swept up".

Chapter 11

Field theory and RG analysis of $A + A \to \emptyset$

It turns out that this is a very simple field theory to analyse. Let us work in the shifted theory :

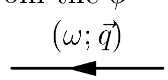
$$S = \int d^d x dt \ \left[\tilde{\phi}[\partial_t \phi] + D_0(\nabla\tilde{\phi})(\nabla\phi) + 2\lambda_0\tilde{\phi}\phi^2 + \lambda_0\tilde{\phi}^2\phi^2\right] \tag{11.1}$$

We can either write this as a stochastic equation :

$$\dot{\phi} = D_0\nabla^2\phi - 2\lambda_0\phi^2 + \zeta \tag{11.2}$$

with $\langle\zeta(x,t)\zeta(x',t')\rangle = -2\lambda_0\phi^2\delta^d(x-x')\delta(t-t')$ and proceed as earlier, or we can write down the Feynman rules by inspection.

The bare propagator, from the $\tilde{\phi}\cdots\phi$ terms, is

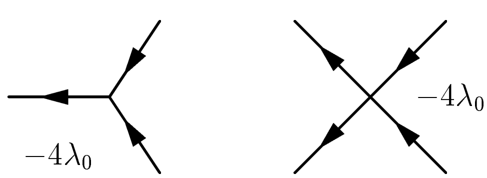


$= \frac{1}{-i\omega + D_0 q^2}$

[Note that we now write this instead of $\frac{1}{-i\omega/D_0 + q^2}$ because

1. The static limit ($\omega \neq 0$) does not correspond to equilibrium statistical mechanics.
2. The coefficient of $-i\omega$ being unity means that particle number is conserved in the absence of reactions.]

We have vertices



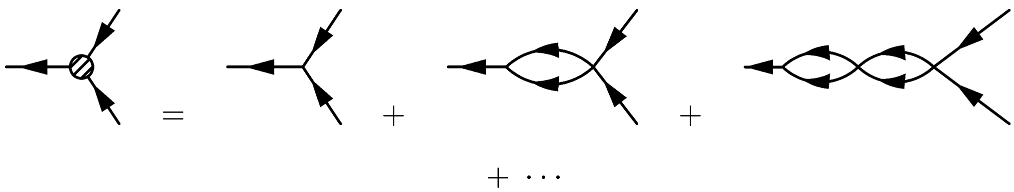
There is an immediate simplification : there are no loop corrections to $G^{(1,1)}$, so :

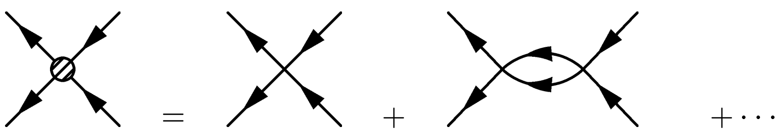
$$\Gamma^{(1,1)} = -i\omega + D_0 q^2 \tag{11.3}$$

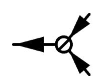
which implies

$$Z_\phi = Z_{\tilde{\phi}} = 1 \quad , \quad Z_D = 1 \tag{11.4}$$

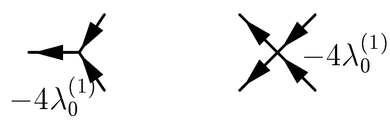
The *only* diagrams renormalising the vertices are :



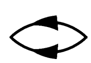


These have a simple physical interpretation :  gives the probability of annihilating given that particles have *not* annihilated in the past.

In principle, we could treat the couplings



as different, in which case we would find

[ $\leftarrow \omega \;\; = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{-i\omega+2Dk^2}$]

$$\lambda_R^{(1)} = \frac{\lambda_0^{(1)}}{1+\frac{4\lambda_0^{(2)}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{-i\omega+2Dk^2}} \tag{11.5}$$

$$\lambda_R^{(2)} = \frac{\lambda_0^{(2)}}{1+\frac{4\lambda_0^{(2)}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{-i\omega+2Dk^2}} \tag{11.6}$$

where the 2's at the denominator of $\frac{4\lambda_0^{(2)}}{2}$ are factors coming from symmetry.

Note that we have to define λ_R at $\omega \neq 0$, otherwise we would have an IR divergence for $d < 2$. This is different from previous case where we could always renormalise in static limit.

Note also that if $\lambda_0^{(1)} = \lambda_0^{(2)}$, then $\lambda_R^{(1)} = \lambda_R^{(2)}$. This is a consequence of probability conservation, $\langle 0|e^{\sum_i a}H(\text{unshifted}) = 0$.

Problem : Consider the processes $A + A \xrightarrow{\lambda_1} \emptyset$ and $A + A \xrightarrow{\lambda_2} A$, and show that the resulting action can be brought to our form by a suitable transformation of the fields ϕ, $\tilde{\phi}$.

We define λ_R at $-i\omega = D\mu^2$ (since D is unrenormalised this is D_0). The integral now gives

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_0^\infty d\alpha \int d^dk\, e^{-2D\alpha k^2 - D\alpha\mu^2} &= \frac{1}{(2\pi)^d}\int_0^\infty d\alpha \left(\frac{\pi}{2D\alpha}\right)^{d/2} e^{-D\alpha\mu^2} \\
&= \frac{1}{(2\pi)^d}\left(\frac{\pi}{2}\right)^{d/2}\Gamma(1-d/2)\frac{1}{D}\mu^{-\epsilon} \\
&\equiv \frac{k_d}{2\epsilon}\frac{\mu^{-\epsilon}}{D} \qquad (11.7)
\end{aligned}$$

$\epsilon = 2 - d$ (note that k_d is regular) and $k_2 = 1/(2\pi)$.

From the action we see that

$$[\tilde{\phi}\phi] = k^d \quad \Rightarrow \quad [\lambda_0]k^d = k^d\omega = [D]k^{d+2} \qquad (11.8)$$

So the dimensionless coupling is

$$g_R = \frac{\lambda_R}{D}\mu^{-\epsilon} \qquad (11.9)$$

which gives

$$g_R = \frac{(\lambda_0/D)\mu^{-\epsilon}}{1 + \frac{k_d}{\epsilon}\frac{\lambda_0}{D}\mu^{-\epsilon}} \qquad (11.10)$$

(exact to all orders !)
Hence

$$\begin{aligned}
\beta(g_R) \equiv \mu\left(\frac{\partial g_R}{\partial\mu}\right)_{\lambda_0,D} &= -\epsilon g_R + \frac{\frac{\lambda_0}{D}\mu^{-\epsilon}\cdot k_d\frac{\lambda_0}{D}\mu^{-\epsilon}}{(1+\frac{k_d}{\epsilon}\frac{\lambda_0}{D}\mu^{-\epsilon})^2} \\
&= -\epsilon g_R + k_d g_R^2 \qquad \text{exact!} \qquad (11.11)
\end{aligned}$$

For $d < 2$ we therefore find an IR fixed point at $g_R^* = \epsilon/k_d \approx 2\pi\epsilon + \mathcal{O}(\epsilon^2)$.

Let us see how to use this to compute the mean density.
In the bare theory, we have $n(t, D, \lambda_0, n_0)$ where n_0 is the initial density. In the renormalised theory, this becomes $n_R(t, D_R, g_R, n_{0R}, \mu)$.
But in fact, $n_R = n$ since there is no field renormalisation. Similarly, $D_R = D$, $n_{0R} = n_0$.
This means we can write down an RG equation :

$$\left(\mu\frac{\partial}{\partial\mu}\right)_{D,\lambda_0,n_0} n_R(t, D, g_R, n_0, \mu) = 0 \tag{11.12}$$

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R}\right] n_R(t, D, g_R, n_0, \mu) = 0 \tag{11.13}$$

Dimensional analysis tells us that

$$n_R(t, D, n_0, \mu) = \mu^d \Phi(\mu^2 Dt, n_0\mu^{-d}) \tag{11.14}$$

so

$$\mu\frac{\partial}{\partial\mu} n_R = \left(d - dn_0\frac{\partial}{\partial n_0} + 2Dt\frac{\partial}{\partial(Dt)}\right) n_R \tag{11.15}$$

and

$$\left[Dt\frac{\partial}{\partial(Dt)} + \frac{1}{2}\beta(g_R)\frac{\partial}{\partial g_R} - \frac{1}{2d}n_0\frac{\partial}{\partial n_0} + \frac{d}{2}\right] n_R = 0 \tag{11.16}$$

The solution of this is :

$$n_R(t, D, g_R, n_0, \mu) = \mu^{-d}(Dt)^{-d/2}\, n_R\left(Dt = \mu^{-2}, n_0 = (mu^2 Dt)^{d/2}, \tilde{g_R}, \mu\right) \tag{11.17}$$

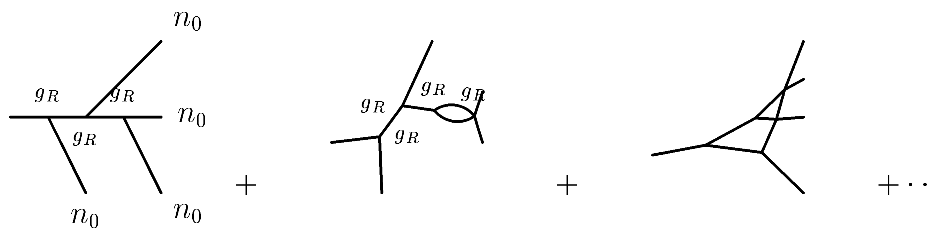
where $\tilde{g_R}(\mu^2 Dt)$ is the running coupling.

As $t \to \infty$, $\tilde{g_R} \to g_R^* = \mathcal{O}(\epsilon)$.

Note that *if* we can ignore the exploding factor $n_0(\mu^2 Dt)^{d/2}$, we have $n \propto (Dt)^{-d/2}$, so the exponent is exact for $d < 2$. (This may be argued on dimensional grounds if n is independent of λ_0 — but these ignore the possibility of anomalous dimensions).

To proceed further, we have to evaluate the RHS of (11.17). Fortunately, we can do this since $g_R^* = \mathcal{O}(\epsilon)$ is small near $d = 2$.

What we do is consider *all* the diagrams for n_R (or n) at a *given order* in n_0 :



To lowest order in $g_R = \mathcal{O}(\epsilon)$ the diagrams are *tree* diagrams.

The sum of these gives the rate equation result. Thus

$$n_R\left(Dt = \mu^{-2}, n_0(\mu^2 Dt)^{d/2}, \epsilon/k_d, \mu\right) = \frac{n_0(\mu^2 Dt)^{d/2}}{1 + 2D\mu^{\epsilon}\frac{\epsilon}{k_d} n_0(\mu^2 Dt)^{d/2}\frac{1}{D\mu^2}} \quad + \quad \text{higher orders in } \epsilon \qquad (11.18)$$

As $t \to \infty$, we see that indeed $n_0(\mu^2 Dt)^{d/2}$ drops out and

$$n_R(t, D, n_0, g_R, \mu) \sim \frac{1}{(Dt)^{d/2}}\mu^{-d}\mu^{2-\epsilon}\frac{k_d}{\epsilon} \quad + \quad \text{higher orders} \qquad (11.19)$$

The μ-dependence disappears as it must.

It takes further work to convince oneself that n_0 drops out to all orders in ϵ, and that

$$n \sim \frac{A}{(Dt)^{d/2}} \qquad (11.20)$$

where the *amplitude* A is *universal* and depends only on ϵ.
To lowest order :

$$A = \frac{1}{4\pi\epsilon} + \mathcal{O}(1) \qquad (11.21)$$

Problem : What happens in $d = 2$?

Chapter 12

Conservation laws — The reaction $A + B \to \emptyset$

Consider now the case of 2 species which undergo the reaction $A + B \to \emptyset$. For convenience, we suppose they have equal diffusivities, and we begin with a random but statistically homogeneous mixture of equal densities n_0.
The chief difference in this system is that the density $n_A - n_B$ is locally conserved. We might expect this to slow the reaction as in model B. But if we write down the hamiltonian :

$$H = H_{diffusion} - \lambda \int d^d x [ab - a^\dagger b^\dagger ab] \tag{12.1}$$

we see that $[\lambda] = k^{2-d}$, so apparently $d_c = 2$ as before.
For $d \geq 2$ we expect the rate equations :

$$\begin{aligned} \dot{a} &= D\nabla^2 a - \lambda ab \\ \dot{b} &= D\nabla^2 b - \lambda ab \end{aligned} \tag{12.2}$$

to be valid. If we look for a solution which is homogeneous, we find $a = b \propto 1/(\lambda t)$ as before.
This is indeed incorrect as it ignores the fluctuations in the *initial state*, which do not disappear as in $A + A \to \emptyset$.
In fact, if we write $\psi \equiv a - b$, it satisfies the diffusion equation

$$\dot{\psi} = D\nabla^2 \psi \tag{12.3}$$

so

$$\psi(x,t) = \int d^d x' \; G_0(t, x - x') \psi(x', 0) \tag{12.4}$$

If a and b have a random initial distribution, then $\psi(x', 0)$ has a distribution with $\overline{\psi(x',0)} = 0$ and $\overline{\psi(x',0)\psi(x'',0)} \propto 2n_0 \delta^{(d)}(x' - x'')$.

Thus $\psi(x,t)$ will have a Gaussian distribution with, in particular :

$$\begin{aligned}\overline{\psi(x,t)^2} = 2n_0 \int d^d x\; G_0(t, x-x')^2 &= 2n_0 \int \frac{d^d k}{(2\pi)^d}\; e^{-2Dtk^2} \\ &= 2n_0 \frac{1}{(2\pi)^d} \left(\frac{\pi}{2Dt}\right)^{d/2} \\ &\equiv \frac{\Delta}{t^{d/2}} \qquad (12.5)\end{aligned}$$

Since $\psi(x,t)$ has a Gaussian distribution $\propto \exp(-\frac{\psi(x,t)^2}{2\overline{\psi^2}})$ we can also compute (for later use)

$$\begin{aligned}\overline{\mid a-b \mid} = \overline{\mid \psi(x,t) \mid} &= \int d\psi \;\; \mid \psi \mid e^{-\psi^2/(2\overline{\psi}^2)} \\ &= \sqrt{\frac{2}{\pi}} (\overline{\psi(x,t)^2})^{1/2} \\ &= \sqrt{\frac{2\Delta}{\pi}} \frac{1}{t^{d/4}} \quad = \quad \frac{(2n_0)^{1/2}}{\pi^{1/2}(8\pi)^{d/4}} \frac{1}{(Dt)^{d/4}} \qquad (12.6)\end{aligned}$$

Note that for $d < 4$ this is *slower* that $1/t$, indicating that it is not possible that

$$\bar{a} \sim \bar{b} \sim 1/t \qquad (12.7)$$

This means that locally, either $a(x) \ll b(x)$ or vice-versa, i.e. there is *segregation.*
Then $\mid a-b \mid = \max(a,b)$, so

$$\bar{a} = \frac{1}{2} \overline{\mid \psi \mid} \propto \frac{1}{t^{d/4}} \qquad (t < 4) \qquad (12.8)$$

Chapter 13

Directed percolation

In the previous example, the steady-state was trivial (but the *approach* to it exhibited interesting universal behaviour).
In order to get a non-trivial steady-state, we need branching processes as well.

Examples are provided by *epidemic processes* : consider a lattice where sites may be *infected* (i.e. occupied by a particle A) or not infected (not occupied). We shall allow multiple occupation, but since the interesting behaviour occurs when the probability of occupation is small, this does not matter.
A given site is occupied (infected) at time $t + \Delta t$ if it or its neighbours were infected at time t, but only with some probability. The disease may just die out locally. Thus the hamiltonian has the form

$$H = -\sum_i (a_i^\dagger - 1) F(a_i^\dagger a_i, \sum_{j\,\text{n.n.}\,\text{i}} a_j^\dagger a_j) \tag{13.1}$$

where the -1 ensures conservation of probability.
A simple form to take for F is

$$F = \lambda_1 \sum_j a_j^\dagger a_j - \lambda_2 \left(\sum_j a_j^\dagger a_j \right)^2 \tag{13.2}$$

where the sum is over all neighbours including i. We expect $\lambda_1 > 0$ and $\lambda_2 > 0$ — this is because i can be infected only once in Δt.

If we now let $a_i^\dagger = 1 + \bar{a}_i$ (i.e. make the shift) we find a variety of terms, all proportional to $\bar{a}_i$. We get an effective diffusion term $\bar{a}_i a_j$ proportional to λ_1 and terms proportional to $-\lambda_1 \bar{a} a$, $-\bar{a}^2 a$ and $+\bar{a} a^2$ (where we have been careful to keep track of the signs). All other terms are later shown to be irrelevant.

Going straight to the field theory, the action is :

$$S = \int dtd^dx\, [\tilde{\phi}\partial_t\phi + D_0\nabla\tilde{\phi}\nabla\phi + r_0\tilde{\phi}\phi + u_1\tilde{\phi}\phi^2 - u_2\tilde{\phi}^2\phi] \tag{13.3}$$

where

$$\begin{aligned} r_0 &\propto -\lambda_1 < 0 \\ u_1 > 0 \quad &, \quad u_2 > 0 \end{aligned} \tag{13.4}$$

If we rewrite this as a stochastic equation, we find

$$\dot{\phi} = D_0\nabla^2\phi + \lambda_1\phi - u_1\phi^2 + \zeta \tag{13.5}$$

where

$$\langle\zeta(x,t)\zeta(x',t')\rangle \propto u_2\phi\delta^{(d)}(x-x')\delta(t-t') \tag{13.6}$$

Ignoring the noise, we see that there are two possible steady-states :

- $\langle\phi\rangle = 0$: the inactive or absorbing state (if the system starts here, it stays here).
- $\langle\phi\rangle = -r_0/u_1$: the active state.

In the rate equation approximation, the latter state is the dynamically stable one for all $r_0 < 0$ (i.e. $\lambda_1 > 0$). But, once the fluctuations are included, there is a non-trivial critical value of $r_{0c} < 0$.

This action is therefore very general and describes a dynamic transition from an inactive state (with no fluctuations) to an active state (with fluctuations) as a function of a control parameter. For historical reasons, it is called the *directed percolation* ("DP") universality class.

In DP, "time" is a discrete space dimension, usually on a lattice (see Fig. 13.1).
It is usual to rescale the fields $\tilde{\phi}$ & ϕ so that the coefficients of $\tilde{\phi}\phi^2$ and $-\tilde{\phi}^2\phi$ are equal. Thus

$$S = \int dtd^dx\, [\tilde{\phi}\partial_t\phi + D_0\nabla\tilde{\phi}\nabla\phi + r_0\tilde{\phi}\phi + \frac{1}{2}u_0\tilde{\phi}\phi^2 - \frac{1}{2}u_0\tilde{\phi}^2\phi] \tag{13.7}$$

In this form, the theory has a remarkable time-reversal symmetry under $t \to -t$, $\phi \to -\tilde{\phi}$, $\tilde{\phi} \to -\phi$. This implies that the renormalised versions of these two couplings will be equal.

The Feynman rules for this theory are quite simple :

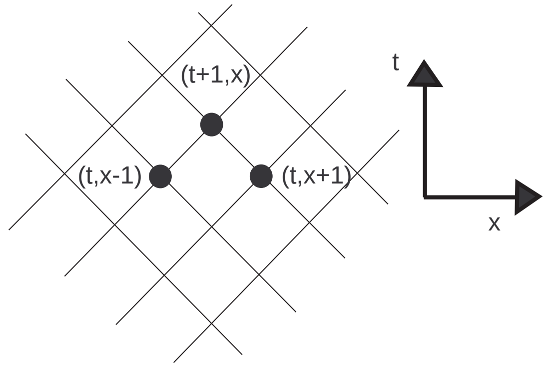
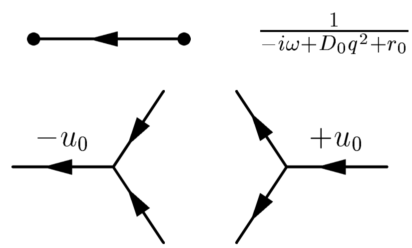
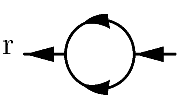


Figure 13.1: (t,x) is occupied with prob. p if either of $(t,x-1)$,$(t,x+1)$ is occupied — directed *site* percolation.

Propagator :

Vertices :



Now, however, there are corrections to the propagator 

To 1-loop :

$$\Gamma^{(1,1)} = -i\omega + D_0q^2 + r_0 - (-u_0)(u_0)\frac{1}{2}\int dk\, \frac{1}{-i\omega + D_0k^2 + D_0(q-k)^2 + 2r_0} + \cdots \tag{13.8}$$

where now $\int dk$ stands for $\int d^dk/(2\pi)^d$.
Notice that the loop corrections act to make $r_R > r_0$ so that at the critical point where $r_R = 0$, $r_{0c} < 0$ as advertised.

Power counting : $[\tilde{\phi}\phi] = k^d$ as usual. Because of the symmetry we choose $[\tilde{\phi}] = [\phi] = k^{d/2}$. Then $[u_0]k^{3d/2} = k^d\omega = k^{d+2}[D_0]$, so $[u_0/D_0] = k^{2-d/2}$ so that the upper critical dimension is $d_c = 4$.

If we study the vertex functions $\Gamma^{(m,n)}$ we find in $d = 4$ that

$$[\Gamma^{(1,1)}] = [D_0]k^2 \quad , \quad [\Gamma^{(1,2)}] = [\Gamma^{(2,1)}] = [D_0]k^0 \tag{13.9}$$

so we have to regularise the following quantities : (mass(r_0) renormalisation assumed done) : $\Gamma^{(1,1)}$, $\frac{\partial\Gamma^{(1,1)}}{\partial k^2}$, $\Gamma^{(1,1)}$, $\frac{\partial\Gamma^{(1,1)}}{\partial(-i\omega)}$ and $\Gamma^{(1,2)} = -\Gamma^{(2,1)}$.

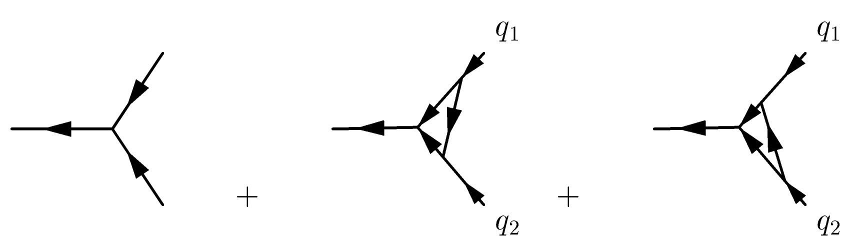
Because of the form of the bare propagator ($-i\omega + D_0 q^2$ rather than $\frac{-i\omega}{D_0} + q^2$), we demand :

1. $\left(\frac{\partial}{\partial(-i\omega)}\Gamma_R^{(1,1)}\right)_{NP} = 1$ to define $Z_\phi = Z_{\tilde{\phi}}$
2. $\left(\frac{\partial}{\partial q^2}\Gamma_R^{(1,1)}\right)_{NP} = D_R \equiv Z_D^{+1} D_0$
3. $u_R = \left(-\Gamma_R^{(1,2)}\right)_{NP} = \left(\Gamma_R^{(2,1)}\right)_{NP}$

so

$$\begin{aligned} Z_\phi^{-1} = \frac{\partial}{\partial(-i\omega)}\Gamma^{(1,1)} &= 1 - \frac{u_0^2}{2}\int\left(\frac{dk}{(D_0k^2 + D_0(q-k)^2)^2}\right)_{q^2=\mu^2} \\ &= 1 - \left(\frac{u_0}{D_0}\right)^2 \frac{1}{2}\frac{1}{4}\frac{2\pi^2}{(2\pi)^4}\frac{\mu^{-\epsilon}}{\epsilon} + \mathcal{O}(u_0^4) \end{aligned} \tag{13.10}$$

$$\begin{aligned} D_R &= Z_\phi \frac{\partial}{\partial q^2}\Gamma^{(1,1)} \\ &= Z_\phi\left[D_0 + \frac{u_0^2}{2}\frac{\partial}{\partial q^2}\int\frac{dk}{D_0(k+\frac{1}{2}q)^2 + D_0(k-\frac{1}{2}q)^2}\right]_{q^2=\mu^2} + \cdots \\ &= Z_\phi D_0\left[1 - \frac{u_0^2}{2}\frac{1}{2}\int\frac{dk}{D_0(k+\frac{1}{2}q)^2 + D_0(k-\frac{1}{2}q)^2} + \cdots\right] \\ &= Z_\phi D_0\left[1 - \left(\frac{u_0}{D_0}\right)^2\frac{1}{2}\frac{1}{2}\frac{1}{4}\frac{2\pi^2}{(2\pi)^4}\frac{\mu^{-\epsilon}}{\epsilon} + \cdots\right] \end{aligned} \tag{13.11}$$

$\Gamma^{(1,2)} =$ 

$$
\begin{aligned}
&= -u_0 + (-u_0)^2 u_0 \int \frac{dk}{[D_0 k^2 + D_0(q_1 - k)^2][D_0 k^2 + D_0(q_1 + q_2 - k)^2]} \\
&\quad + (\mathrm{q}_1 \leftrightarrow \mathrm{q}_2) \text{ term} \\
&= -u_0 + 2\frac{u_0^3}{D_0^2}\frac{1}{4}\frac{2\pi^2}{(2\pi)^4}\frac{\mu^{-\epsilon}}{\epsilon} \qquad (13.12)
\end{aligned}
$$

$$
u_R = -\Gamma_R^{(1,2)} = -Z_\phi^{3/2}\Gamma^{(1,2)} \qquad (13.13)
$$

Finally, the dimensionless coupling is

$$
\begin{aligned}
g_R &= (u_R/D_R)\mu^{-\epsilon/2} \\
&= \frac{(u_0/D_0)\mu^{-\epsilon/2}[1 - \frac{1}{2}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon}]}{[1 - \frac{1}{16}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon}][1 - \frac{1}{16}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon} + \cdots]} \\
&= (u_0/D_0)\mu^{-\epsilon/2}[1 - \frac{3}{8}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon} + \cdots] \qquad (13.14)
\end{aligned}
$$

$$
\beta(g_R) = \frac{-\epsilon}{2}g_R + \frac{3}{8}K_4 g_R^3 + \mathcal{O}(g_R^5) \qquad (13.15)
$$

As usual, we have a non-trivial IR fixed point, this time with

$$
g_R^{*2} = \frac{4\epsilon}{3} + \cdots \qquad (13.16)
$$

$$
Z_D = \left(1 + \frac{1}{8}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon}\right)\left(1 - \frac{1}{16}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon}\right) \qquad (13.17)
$$

so

$$
\gamma_D = -\frac{1}{16}K_4 g_R^2 + \cdots \quad \Rightarrow \quad z = 2 - \frac{\epsilon}{12} + \mathcal{O}(\epsilon^2) \qquad (13.18)
$$

$$
Z_\phi = 1 + \frac{1}{8}K_4\frac{(u_0/D_0)^2\mu^{-\epsilon}}{\epsilon} + \cdots \qquad (13.19)
$$

$$
\gamma_\phi = -\frac{1}{8}K_4 g_R^2 + \cdots \qquad (13.20)
$$

13.0.1 Scaling behaviour

$$
\mu\left(\frac{\partial}{\partial\mu}\right)_{u_0, D_0, \cdots} Z_\phi^{-1}\Gamma_R^{(1,1)} = 0 \qquad (13.21)
$$

$$\left[\mu\frac{\partial}{\partial\mu} - \underbrace{\frac{1}{Z_\phi}\mu\frac{\partial}{\partial\mu}Z_\phi}_{-\gamma_\phi} + \beta(g_R)\frac{\partial}{\partial g_R} + \underbrace{\mu\frac{\partial D_R}{\partial\mu}\frac{\partial}{\partial D_R}}_{-\gamma_D D_R\frac{\partial}{\partial D_R}}\right]\Gamma_R^{(1,1)} = 0 \qquad (13.22)$$

$$\gamma_\phi = \mu\frac{\partial}{\partial\mu}\ln Z_\phi \qquad \gamma_D = \mu\frac{\partial}{\partial\mu}\ln Z_D \qquad (13.23)$$

At the fixed point :

$$\left[\mu\frac{\partial}{\partial\mu} - \gamma_\phi^* - \gamma_D^* D_R\frac{\partial}{\partial D_R}\right]\Gamma_R^{(1,1)} = 0 \qquad (13.24)$$

13.0.2 Dimensional analysis

$$\Gamma_R^{(1,1)} = D_R\mu^2\Phi\left[\frac{k}{\mu}, \frac{\omega}{(D_R k^2)}\right] \qquad (13.25)$$

so

$$D_R\frac{\partial}{\partial D_R} = 1 - \omega\frac{\partial}{\partial\omega} \qquad (13.26)$$

$$\mu\frac{\partial}{\partial\mu} + k\frac{\partial}{\partial k} = 2 - 2\omega\frac{\partial}{\partial\omega} \qquad (13.27)$$

Hence

$$\left[-k\frac{\partial}{\partial k} + 2 - \gamma_\phi^* - (2+\gamma_D^*)\omega\frac{\partial}{\partial\omega}\right]\Gamma_R^{(1,1)} = 0 \qquad (13.28)$$

$$\Gamma^{(1,1)}(\omega, k) = k^{2-\gamma_\phi^*+\gamma_D^*}\,\Phi\left(\frac{\omega}{k^{2+\gamma_D^*}}\right) \qquad (13.29)$$

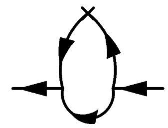
and we end up with dynamic scaling again.

As $k \to 0$ we expect $\Gamma^{(1,1)}(\omega, 0) \sim \omega^{1-\gamma_\phi^*/(2+\gamma_D^*)}$ implying that the density of infected sites decays as $\int d\omega\, e^{i\omega t}\Gamma^{(1,1)-1} \sim t^{-\gamma_\phi^*/(2+\gamma_D^*)}$ and that the dynamic exponent takes the value

$$z = 2 + \gamma_D^* \qquad (13.30)$$

13.0.3 Away from the critical point

In DP we have a control parameter r_0, like a bare mass. There is another critical exponent associated with this, which may be found by studying the renormalisation of the composite operator $\tilde{\phi}\phi$. The 1-loop diagram is



Denoting $\Delta_0 =\mid r_0 - r_{0c} \mid$ we find the scaling behaviour

$$\Gamma_R^{(1,1)}(\omega, k, \Delta_0) = k^{2-\gamma_\phi^*-\gamma_D^*}\Phi(\omega/k^z, k/\Delta_0^{\nu_\perp}) \tag{13.31}$$

which may be rewritten in various other ways, e.g. $\omega/\Delta_0^{\nu_\parallel=2\nu_\perp}$.

For $r_0 > r_{0c}$, $G^{(1,1)}$ decays exponentially like $e^{-t/\tau}$ with $\tau \propto \Delta_0^{-\nu_\parallel}$.

For $r_0 < r_{0c}$, starting from a single infected site we go to a finite density in the active state. In that case

$$G^{(1,1)}(t,x) \stackrel{t\to\infty}{\longrightarrow} p(\mid \Delta_0 \mid) \tag{13.32}$$

Replacing now $G^{(1,1)}$ and p by their complete expressions, we get

$$\begin{aligned} \int d\omega d^d k\ k^{-2+\gamma_\phi^*+\gamma_D^*}\Phi\left(\frac{\omega}{\mid \Delta_0 \mid^{\nu_\parallel}}, \frac{k}{\mid \Delta_0 \mid^{\nu_\perp}}\right) &\propto \mid \Delta_0 \mid^{\nu_\parallel + d\nu_\perp - (2-\gamma_\phi^*-\gamma_D^*)\nu_\perp} \\ &= \mid \Delta_0 \mid^{(d+\gamma_\phi^*)\nu_\perp} \\ &= \mid \Delta_0 \mid^{\beta} \end{aligned} \tag{13.33}$$

defining the "order parameter" exponent β.

Chapter 14

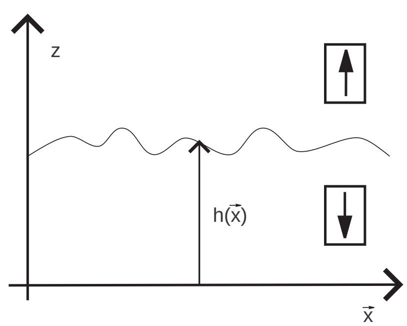
The Kardar-Parisi-Zhang equation

This was originally formulated as a model of a growing interface, but it can also be mapped to :

- the "noisy" Burgers equation in hydrodynamics.
- directed polymers in a random medium.

Consider an Ising model below T_c, with an interface between $\uparrow$ and $\downarrow$ phases. We use a continuus spin $S = S(\vec{x}, z)$

$$\mathcal{H} = \int [\frac{1}{2}(\nabla S)^2 + V(S)]\, d^d x dz \tag{14.1}$$



The position of the interface is $z = h(\vec{x})$.

In equilibrium we find a flat solution $h = const$, $S = f(z-h)$ by minimising $\mathcal{H}$, so

$$f''(z) = V(f(z)) \tag{14.2}$$

Let this solution be $f(\cdot)$.
When the interface fluctuates, we assume that its profile does not vary, only its position and angle. So we write :

$$S(\vec{x}, z, t) = f\left(\frac{z - h(\vec{x}, t)}{\sqrt{1 + (\vec{\nabla}_\perp h)^2}}\right) \tag{14.3}$$

where $\vec{\nabla}_\perp$ is the derivative in the $\vec{x}$-directions (note that d is now the number of *transverse* dimensions).
We now add a magnetic field : $\mathcal{H} \longrightarrow \mathcal{H} + \mu \int S\, d^d x dz$ which will *drive* the interface (i.e. make it move in the z-direction), and write down model A :

$$\dot{S} = D\left(\frac{\partial^2 S}{\partial z^2} + \vec{\nabla}_\perp^2 S - V'(S) - \mu\right) + \zeta \tag{14.4}$$

Inserting the Ansatz (14.3) :

$$\frac{-\dot{h}}{\sqrt{\cdot}} f'\left(\frac{z-h}{\sqrt{\cdot}}\right) = D\left(\frac{1}{(\sqrt{\cdot})^2} f'' - \frac{\vec{\nabla}_\perp^2 h f'}{\sqrt{\cdot}} + \frac{(\vec{\nabla}_\perp h)^2 f''}{(\sqrt{\cdot})^2} - V'(f) - \mu\right) + \zeta \tag{14.5}$$

where we have ignored some terms with ≥ 3 derivatives.
The first, third and fourth terms in the brackets cancel, because f satisfies (14.2.)
We multiply this equation by $f'(\frac{z-h}{\sqrt{\cdot}})$ and integrate $\int_{-\infty}^{\infty} dz$:

$$\dot{h} \int_{-\infty}^{\infty} f'(u)^2\, du = D\nabla_\perp^2 h \int_{-\infty}^{\infty} f'(u)^2\, du + D\mu\sqrt{\cdot} \int_{-\infty}^{\infty} f'(u)\, du + \tilde{\zeta} \tag{14.6}$$

where

$$\tilde{\zeta}(\vec{x}, t) = \sqrt{\cdot} \int_{-\infty}^{\infty} \zeta(\vec{x}, z, t) f'(z)\, dz \tag{14.7}$$

Expanding out $\sqrt{\cdot} \approx 1 + \frac{1}{2}(\nabla_\perp h)^2 + \cdots$, we finally obtain an equation of the form :

$$\dot{h} = v + \frac{1}{2}\lambda(\nabla_\perp h)^2 + \nu \nabla_\perp^2 h + \eta \tag{14.8}$$

where $v \propto D\mu$, $\lambda \propto D\mu$, $\nu \propto D$ and $\langle \eta(\vec{x}, t)\eta(\vec{x}', t')\rangle = 2D\delta^{(d)}(\vec{x} - \vec{x}')\delta(t - t')$.

We can remove v by going to a moving frame $h \longrightarrow h' = h - vt$. This gives the KPZ equation. Note that we have lost detailed balance — the rhs cannot be written as $-D\frac{\delta F}{\delta h} + \eta$.

14.1 KPZ equation : response function formalism

Action :

$$\int dt d^d x \, [\tilde{h}(\dot{h} - \frac{1}{2}\lambda(\nabla h)^2 - \nu \nabla^2 h) - D\tilde{h}^2] \tag{14.9}$$

Dimensional analysis :

$$[\tilde{h}h] = k^d, \quad [\nu] = \omega k^{-2}, \quad [\lambda][\tilde{h}h^2] = k^{d-2}\omega, \quad [D][\tilde{h}^2] = k^d\omega, \quad \text{so}$$

$$[\lambda^2 D](\tilde{h}h)^4 = k^{2d-4}\omega^2 k^d \omega = [\nu]^3 k^{3d+2}$$

Dimensionless expansion parameter is $[\lambda^2 D/\nu^3] = k^{2-d}$:

$$d_c = 2$$

In $d > 2$: the non-linearity $(\nabla h)^2$ is irrelevant (for λ small).
We are therefore lead to the Edwards-Wilkinson theory :

$$\dot{h} = \nu \nabla^2 h + \zeta \tag{14.10}$$

(which satisfies the detailed balance condition with $D = \nu k T$).

In general :
Dynamic scaling : $\langle h(\vec{x},t)h(0,0)\rangle = \mid \vec{x} \mid^{2\chi} \Phi\left(\frac{t}{|\vec{x}|^z}\right)$

- $\chi > 0 \quad \Rightarrow \quad$ interface rough
- $\chi < 0 \quad \Rightarrow \quad$ interface smooth

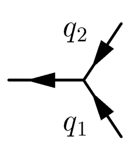
In EW theory, $\langle h(\vec{x},0)h(0,0)\rangle = \int \frac{d^d q}{q^2} e^{iqx} \propto \mid x \mid^{-d+2}$ so

$$\chi_{EW} = 1 - d/2 \qquad \text{smooth for } d > 2. \tag{14.11}$$

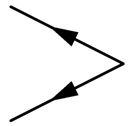
$$z_{EW} = 2 \tag{14.12}$$

14.2 Renormalisation for $d < 2$

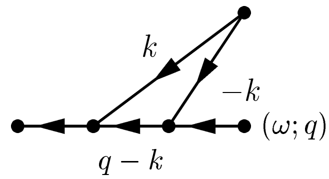
Propagator :  $\langle h\tilde{h}\rangle = \frac{1}{-i\omega + \nu k^2}$

Vertex $\tilde{h}(\nabla h)^2$: 

$\propto \vec{q_1} \cdot \vec{q_2}$

Noise vertex : 

1-loop correction to propagator :



$$\int \frac{d^dk}{(2\pi)^d} \frac{[k\cdot(q-k)][-k\cdot q]}{(-i\omega+\nu k^2+\nu(q-k)^2)(2\nu k^2)} \propto q^2 \int \frac{d^dk}{[k^2]} \tag{14.13}$$

Hence $\frac{\partial}{\partial(-i\omega)}\Gamma^{(1,1)}$ is finite in d=2 (but $\frac{\partial}{\partial q^2}\Gamma^{(1,1)}$ is not).
There is therefore no field renormalisation. We should only renormalise ν, D, λ.
In addition, there is in fact no renormalisation of λ due to *Galilean invariance* : If $\vec{u} = \vec{\nabla}h$:

$$\begin{aligned} \dot{\vec{u}} &= \vec{\nabla}(\frac{\lambda}{2}\vec{u}^2 + \nu\vec{\nabla}u) + \vec{\nabla}\zeta \\ &= \lambda(\vec{u}\cdot\vec{\nabla})\vec{u} + \nu\nabla^2\vec{u} + \vec{\nabla}\zeta \end{aligned} \tag{14.14}$$

If $\lambda = -1$, we can take the first term onto the left-hand side, giving

$$\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u} \equiv \frac{D\vec{u}}{Dt} \tag{14.15}$$

is the convective derivative, as for a fluid : this then gives the noisy Burgers equation (which is the Navier-Stokes equation in the absence of vorticity). For this, we expect Galilean invariance.
For general λ, we in fact have invariance under

$$\vec{x} \longrightarrow \vec{x} - \lambda\vec{v}t \quad , \quad \vec{u} \longrightarrow \vec{u}' = \vec{u}(\vec{x}+\lambda\vec{v}t) + \vec{v} \tag{14.16}$$

(where $\vec{v} =$ const). This reflects the *tilt-invariance* of the original interface model.
Since λ is a *parameter* in this transformation, it can't be renormalised !
So we are left with

$$\begin{aligned} \gamma_D &\equiv \frac{1}{D_R}\mu\frac{\partial}{\partial\mu}D_R \\ \gamma_\nu &\equiv \frac{1}{\nu_R}\mu\frac{\partial}{\partial\mu}\nu_R \end{aligned} \tag{14.17}$$

as non-trivial renormalisation group functions.

Dimensionless coupling :

$$g_R = \frac{\lambda_R^2 D_R}{\nu_R^3}\mu^{d-2} \tag{14.18}$$

so

$$\beta(g_R) \equiv \mu\frac{\partial}{\partial\mu}g_R = g_R(d-2+\gamma_D-3\gamma_\nu) \tag{14.19}$$

Hence, at any *non-trivial* fixed point

$$\gamma_D^* - 3\gamma_\nu^* = 2-d \tag{14.20}$$

As usual,

$$z = 2+\gamma_\nu^* \tag{14.21}$$

The RG equation for $G = \int d^dx\, e^{ik(x-x')}\langle h(x,t)h(x',t')\rangle$ is :

$$\left[\mu\frac{\partial}{\partial\mu} + \gamma_D^* D\frac{\partial}{\partial D} + \gamma_\nu^*\nu\frac{\partial}{\partial\nu}\right]G = 0 \qquad \text{at fixed point} \tag{14.22}$$

$$G = \frac{D}{\nu k^2}\Phi(\frac{\mu}{k}) \tag{14.23}$$

$$\left[-2 - k\frac{\partial}{\partial k} + \gamma_D^* - \gamma_\nu^*\right]G = 0 \tag{14.24}$$

$$G \propto \frac{1}{k^{2-\gamma_D^*+\gamma_\nu^*}} \tag{14.25}$$

$$\chi = 1 - \frac{d}{2} + \frac{\gamma_\nu^* - \gamma_D^*}{2} \tag{14.26}$$

Hence

$$z + \chi = 2 \tag{14.27}$$

This is a remarkable scaling relation which comes from Galilean invariance and the lack of field renormalisation.

14.3 Exact exponents for $d = 1$

Edwards-Wilkinson linear theory satisfies detailed balance :

$$\begin{aligned} \dot{h} &= \nu\nabla^2 h + \zeta \qquad \langle\zeta\zeta\rangle = 2D\delta(.)\delta(.) \\ &= -\nu\frac{\delta}{\delta h}\left[\int \frac{1}{2}(\nabla h)^2\, d^dx\right] + \zeta \end{aligned} \tag{14.28}$$

Equilibrium distribution :

$$P[h] = \exp\left(-\frac{1}{kT_{\rm eff}}\int \frac{1}{2}(\nabla h)^2\, d^dx\right) \tag{14.29}$$

where $kT_{\rm eff} = D/\nu$.
In $d=1$, adding the non-linearity does not affect this !

$$P[h_0]_{\rm eq} = \langle \delta(h - h_0(x,t))\rangle = \int \mathcal{D}h\, \delta_f(h-h_0(x,t))\exp(-\mathcal{H}_{\rm eff}[h]) \tag{14.30}$$

$$\begin{aligned} \frac{d}{dt}P[h_0] &= -\int \mathcal{D}h \sum_x \delta'(h-h_0(x,t))\dot{h}_0 \prod_{x'\neq x}\delta(h-h_0)\exp(-\mathcal{H}_{\rm eff}) \\ &= -\int d^dx \int \mathcal{D}h\, \delta_f(h-h_0)\dot{h}_0 \left(\frac{\delta\mathcal{H}_{\rm eff}}{\delta h}\right)_{h=h_0} \exp(-\mathcal{H}_{\rm eff}) \\ &= \int d^dx\, \langle \frac{1}{2}\lambda(\nabla h)^2\nabla^2 h\rangle + \cdots \end{aligned} \tag{14.31}$$

In general, this is non-zero, but in $d=1$

$$\left(\frac{\partial h}{\partial x}\right)^2\left(\frac{\partial^2 h}{\partial x^2}\right) = \frac{1}{3}\frac{\partial}{\partial x}\left[\left(\frac{\partial h}{\partial x}\right)^3\right] \tag{14.32}$$

giving a total derivative which integrates to zero.
We know the steady-state for $d=1 \Rightarrow$ FDT $\Rightarrow \gamma_D^* = \gamma_\nu^*$ giving

$$z = 3/2 \quad , \quad \chi = 1/2 \tag{14.33}$$

The interface is rough !

14.4 Directed polymer representation

$$\dot{h} = \nu\nabla^2 h + \frac{1}{2}\lambda(\nabla h)^2 + \zeta \tag{14.34}$$

Let now

$$h = \frac{2\nu}{\lambda}\ln w \qquad \text{(Cole -- Hopf transformation)} \tag{14.35}$$

then

$$\frac{2\nu}{\lambda}\frac{\dot{w}}{w} = \frac{2\nu^2}{\lambda}\left[\frac{\nabla^2 w}{w} - \frac{(\nabla w)^2}{w^2}\right] + \frac{2\nu^2}{\lambda}\frac{(\nabla w)^2}{w^2} + \zeta \tag{14.36}$$

$$\dot{w} = \nu\nabla^2 w + \frac{\lambda}{2\nu}w\zeta \qquad \text{linear!} \tag{14.37}$$

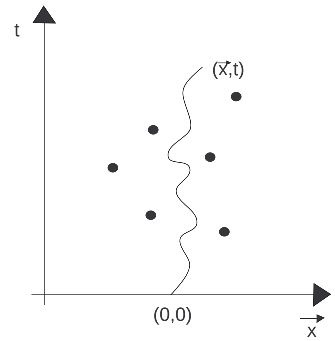


Figure 14.1: A polymer in a random medium (dots represent impurities)

Polymer in a random medium (see Fig. 14.1):
Let $w(\vec{x},t)$ be the partition function given the ends are at $(0,0)$ and $(\vec{x},t)$.

$$w(\vec{x},t) = \int_{\vec{x}'(0)=0}^{\vec{x}'(t)=\vec{x}} \mathcal{D}\vec{x}' \, \exp\left(-\frac{1}{kT}\int_0^t dt' \, [\frac{\epsilon}{2}(\frac{d\vec{x}'}{dt'})^2 + V(\vec{x}',t')]\right) \tag{14.38}$$

with V being a random potential. This is like a Feynman path integral, so w obeys a "Schrödinger" equation :

$$T\frac{\partial w}{\partial t} = \frac{T^2}{2\epsilon}\nabla^2 w + Vw \tag{14.39}$$

which is the same equation as (14.37) with $\nu = \frac{T}{2\epsilon}$, and $\zeta = \frac{1}{\epsilon\lambda}V$

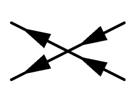
14.4.1 Renormalisation for $d > 2$

Response function formalism :

$$\int d^d x dt \left[\tilde{w}(\dot{w} - \nu\nabla^2 w) - \frac{\lambda^2 D}{2\nu^2}\tilde{w}^2 w^2\right] \tag{14.40}$$

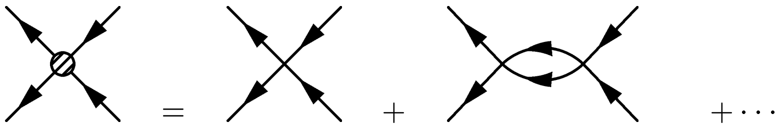
Feynman rules :

$\frac{1}{-i\omega+\nu k^2}$



$+\frac{\lambda^2 D}{\nu^2}$

Renormalisation is simple : just as in $A + A \longrightarrow \emptyset$:

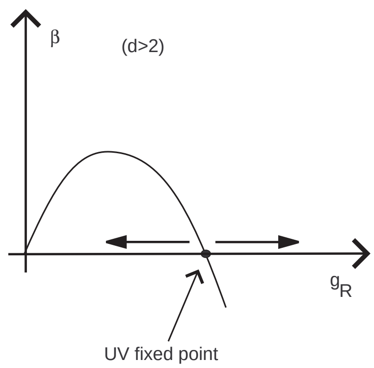


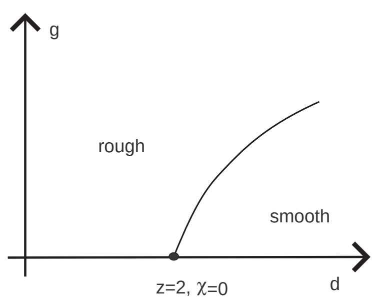
but the coupling constant has now a different sign.

$$\beta(g_R) = -g_R[2-d] - bg_R^2 \tag{14.41}$$

To all orders :

$$g_R^* = (d-2)/b \tag{14.42}$$





The interpretation of this is a *roughening transition* at $g = g^*$.

Exactly at the transition, we have

$$\nu_\perp = \frac{1}{d-2} \tag{14.43}$$

where

$$\xi_{\text{smooth}} \propto (g^* - g)^{-\nu_\perp} \tag{14.44}$$

Unsolved problems :

1. What is the nature of the rough (strong coupling) phase for $d \geq 2$?
2. Is there an upper critical dimension ? [For $d > 4$, $\nu_\perp = \frac{1}{d-2}$ violates the rigorous inequality $d\nu_\perp > 2$]